Uniformly Chaotic Finite-Range Lattice Models
And the Characterisation of the Set of Ground States Thereof

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Thermodynamic Formalism and Chaoticity

Controlling Markers Distribution

Building an Appropriate Structure
  Turing Machines as Tilings
  The Robinson Tiling(s)
  Structure for Entropy Control

Forcing Complex Structures
Thermodynamic Formalism and Chaoticity
Gibbs Measures on Finite Spaces

- $\Omega$ a finite set of states.
- $E : \Omega \to \mathbb{R}^+$ an energy function.
- $\beta$ the inverse temperature.

**Theorem (Variational Principle)**

The distribution $\mu_\beta(\omega) \propto \exp(-\beta E(\omega))$ is the only maximiser of $\mu \mapsto H(\mu) - \beta \mu(E)$, with $H(\mu) := \sum - \log_2(\mu(\omega)) \mu(\omega)$ the entropy.

We call $\mu_\beta$ a Gibbs measure.
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We call \( \mu_{\beta} \) a Gibbs measure.

- At high temperatures, as \( \beta \to 0 \), we converge to the uniform distribution \( \mathcal{U}(\Omega) \), that maximises \( H \).
- At low temperatures, as \( \beta \to \infty \), we converge to the uniform distribution \( \mathcal{U}(\Omega^*) \), that maximises \( H \) among measures of minimal energy, supported by \( \Omega^* := \text{arg min}(E) \).
Invariant Gibbs Measures on Lattice Models

- \( \Omega_A := A^{\mathbb{Z}^d} \) the phase space, with \( A \) a finite alphabet.
- \( \mathbb{Z}^d \overset{\sigma}{\looparrowright} \Omega_A \) the shift action, so that \( \sigma^x(\omega)_y = \omega_{y-x} \) for any \( x, y \in \mathbb{Z}^d \) and \( \omega \in \Omega_A \).
- \( \mathcal{M}_\sigma(\Omega_A) \) the set of invariant measures on \( \Omega_A \), such that \( \mu \circ \sigma^x = \mu \) for any \( x \in \mathbb{Z}^d \).
- \( \varphi : \Omega_A \to \mathbb{R}^+ \) a continuous potential, the contribution of \( 0 \in \mathbb{Z}^d \) to the energy.
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- $\varphi : \Omega_A \rightarrow \mathbb{R}^+$ a continuous potential, the contribution of $0 \in \mathbb{Z}^d$ to the energy.

**Definition (Pressure Function)**

Define the pressure $p_\mu(\beta) := h(\mu) - \beta \mu(\varphi)$, with $h(\mu) := \lim \frac{1}{n} H(\mu_{[0,n-1]^d})$ the entropy per site.

Let $\mathcal{G}_\sigma(\beta) := \arg \max_{\mu \in \mathcal{M}_\sigma} p_\mu(\beta)$ the set of Gibbs measures.

- $\varphi$ has finite range if it is *locally constant*, if $\varphi(\omega)$ only depends on $\omega_{[-r,r]^d}$. 
Limit Behaviour for Ground States

- We call $(\mu_\beta \in \mathcal{G}_\sigma(\beta))_{\beta > 0}$ a cooling trajectory of the model.
- Denote $\mathcal{G}_\sigma(\infty) := \text{Acc}_{\beta \to \infty} \mathcal{G}_\sigma(\beta)$ the set of ground states, of accumulation points of all the cooling trajectories.
- $\mathcal{G}_\sigma(\infty)$ is a connected compact set (for the weak-* topology).
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**Lemma**

Assume that \(X := \{\omega \in \Omega_\mathcal{A}, \forall x \in \mathbb{Z}^d, \varphi \circ \sigma^x(\omega) = 0\} \neq \emptyset\). Then \(\mathcal{G}_\sigma(\infty) \subset \mathcal{M}_\sigma(X)\), and the ground states have maximal entropy \(h\) in \(\mathcal{M}_\sigma(X)\).

- Measures that maximise \(h\) in \(\mathcal{M}_\sigma(X)\) are not necessarily in \(\mathcal{G}_\sigma(\infty)\).

What can we ask about \(\mathcal{G}_\sigma(\infty)\)?
Stability and Chaos

**Definition (Stability)**
A model is stable if all the cooling trajectories converge to the same limit.

**Definition (Chaoticity)**
A model is chaotic if there is no converging cooling trajectory.

**Definition (Uniformity)**
A model is uniform if all the cooling trajectories have the same accumulation set.
Recap of Behaviours

Chaoticity:

\[ \forall \nu, \forall (\mu_\beta), \mu_\beta \not\rightarrow \nu \]

Stability:

\[ \exists \nu, \forall (\mu_\beta), \mu_\beta \rightarrow \nu \]

**Figure 1:** Inventory and comparison of model behaviours.
Recap of Behaviours

Chaoticity:
\[ \forall \nu, \forall (\mu_\beta), \mu_\beta \not\rightarrow \nu \]

Uniformity:
\[ \forall (\mu_\beta), \text{Acc}(\mu_\beta) = G_{\sigma}(\infty) \]

Stability:
\[ \exists \nu, \forall (\mu_\beta), \mu_\beta \rightarrow \nu \]

**Figure 1:** Inventory and comparison of model behaviours.
Recap of Behaviours

Chaoticity:

∀ν, ∀(μβ), μβ ≠ ν

Uniformity:

∀(μβ), Acc(μβ) = Gσ(∞)

Stability:

∞ ← |Gσ(∞)| → 1

∃ν, ∀(μβ), μβ → ν

Figure 1: Inventory and comparison of model behaviours.
## Current Knowledge

### Lemma

*A one-dimensional finite range model induces a stable model.*

### Theorem (Chazottes and Hochman 2010)

*There exists a one-dimensional potential inducing a chaotic model.*

*There exists a three-dimensional finite range potential inducing a chaotic model.*

### Theorem (Chazottes and Shinoda 2020; Barbieri et al. 2022)

*There exists a two-dimensional finite range potential inducing a chaotic model.*
Realisation Result on the Limit Set

- Remind that $G_\sigma(\infty)$ must be connected.
- When $\varphi$ is a computable potential inducing a uniform model, $G_\sigma(\infty)$ must be a $\Pi_2$-computable set.

**Theorem (Gayral, Sablik, and Taati 2023)**

There exists a class of two-dimensional finite range potentials, inducing uniform models both stable and chaotic.

More precisely, we can realise any connected $\Pi_2$-computable compact set $X$ as $G_\sigma(\infty)$, up to a fixed computable affine homeomorphism.
Controlling Markers Distribution
General Idea for Chaoticity

We have two measures $\mu \neq \mu'$ s.t. $d(\mu, \mu') \geq r$ and:

$$d(G_\sigma(\beta), \mu) \leq \frac{r}{3}$$

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**Figure 2:** Alternating between mutually exclusive adherence values on non-overlapping intervals.
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**Figure 2:** Alternating between mutually exclusive adherence values on non-overlapping intervals.

Thus $\text{Acc}(\mu_\beta)$ intersects the disjoint neighbourhoods of both $\mu$ and $\mu'$. 
General Idea for Uniformity

We want \((\mu_n)\) and \(\varepsilon_n \to 0\) s.t.:

\[
d (G_\sigma(\beta), \mu_1) \leq \varepsilon_1 \quad \text{and} \quad d (G_\sigma(\beta), \mu_3) \leq \varepsilon_3
\]

\[
d (G_\sigma(\beta), \mu_2) \leq \varepsilon_2 \quad \text{and} \quad d (G_\sigma(\beta), \mu_4) \leq \varepsilon_4
\]

\textbf{Figure 3}: Contracting tube of measures with overlapping intervals.
General Idea for Uniformity

We want \((\mu_n)\) and \(\varepsilon_n \to 0\) s.t.:

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d(\mathcal{G}_\sigma(\beta), \mu_2) \leq \varepsilon_2 \quad d(\mathcal{G}_\sigma(\beta), \mu_4) \leq \varepsilon_4
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**Figure 3**: Contracting tube of measures with overlapping intervals.

Thus \(\text{Acc}(\mu_\beta) = \mathcal{G}_\sigma(\infty) = \text{Acc}(\mu_n)\).
From Thermodynamics to Combinatorics

- $\mathcal{F}$ a finite set of forbidden patterns $w \in \mathcal{A}^{|w|}$, each on a finite window $l(w) \in \mathbb{Z}^d$.
- $p \in \mathcal{A}^l$ is locally admissible if no translation of a forbidden pattern occurs within it.
- $\mathcal{F}$ induces a subshift of finite type (SFT) $X_{\mathcal{F}} \subset \Omega_{\mathcal{A}}$, closed and shift-invariant, made of configurations that are globally admissible.

**Example**

In one dimension, let $\mathcal{A} = \{0, 1\}$ and $\mathcal{F} = \{100, 101\}$. Then:

- $0^\mathbb{Z} \in X_{\mathcal{F}}$, $1^\mathbb{Z} \in X_{\mathcal{F}}$, $\cdots 000111 \cdots \in X_{\mathcal{F}}$,
- $10$ is locally admissible, but doesn’t occur in any $\omega \in X_{\mathcal{F}}$. 

*Lemma*

Assume that $X_{\mathcal{F}} \neq \emptyset$, and let $\phi := 1^\mathcal{F}$ covers $\mathcal{A}$ the induced finite range potential. Then $G_{\sigma}(\infty) \subset M_{\sigma}(X_{\mathcal{F}})$, and the ground states have maximal entropy $h$ in $M_{\sigma}(X_{\mathcal{F}})$. 

From Thermodynamics to Combinatorics

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**Lemma**

Assume that $X_\mathcal{F} \neq \emptyset$, and let $\varphi := 1_\mathcal{F}^{\text{covers}} 0$ the induced finite range potential. Then $G_\sigma(\infty) \subset \mathcal{M}_\sigma(X_\mathcal{F})$, and the ground states have maximal entropy $h$ in $\mathcal{M}_\sigma(X_\mathcal{F})$. 
Control of Markers on a Temperature Interval

**Definition (Marker Set with Margin Factor $\tau$)**

A marker set $Q \subset \mathcal{A}^{l_{\ell}}$ (with $l_{\ell} := [0, \ell - 1]^d$) is made of non-overlapping patterns, s.t. any locally admissible $\omega \in \mathcal{A}^{(2+\tau)\ell - 1}$ must contain a marker somewhere.

**Theorem (Adapted from Chazottes and Hochman 2010)**

Denote $G_n$ the locally admissible tilings of $I_n$, and $\mu_Q$ the cond. measure of $\mu$ on $Q$. We have constants $C, C'$ s.t. for any marker set $Q$ and $\varepsilon, \kappa > 0$, if

$$\frac{\log_2(\#G_n)}{\#I_n} \geq (1 - \kappa) \frac{\log_2(\#Q)}{\#I_{\ell}}$$

and

$$\beta \in \left[ C \frac{\#I_{\ell}}{\varepsilon}, C' n \varepsilon \right]$$

then, for any $\mu \in G_0(\beta)$:

$$\mu (Q \text{ covers } 0) = 1 - O(\varepsilon + \tau) \quad \text{ and } \quad H (\mu_Q) \geq (1 - 2\kappa) \log_2(\#Q) - H(\kappa) - O(\varepsilon + \tau).$$
Building an Appropriate Structure
(aka LEGO for Grownups)

Turing Machines as Tilings
Turing Machines

Turing machines are a model equivalent to real-life computers and algorithms.

Formally, $M$ is made of:

- internal states $Q$,
- an initial state $q_0 \in Q$,
- accepting states $Q_A \subset Q$,
- rejecting states $Q_R \subset Q$,
- an input alphabet $\mathcal{A}$,
- a tape alphabet $\Gamma \supset \mathcal{A} \sqcup \{\#\}$,
- a transition function $\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$.

Figure 4: Real-life Turing machine
(Source: wikimedia.org)
A Turing machine $M = (Q, q_0, Q_A, Q_R, \mathcal{A}, \Gamma, \delta)$ can be simulated by a Wang tileset:

Figure 5: Turing space-time diagram Wang tiles for each letter $a \in \Gamma$. 

\[ \delta(q, a) = (q', b, D) \]

\[ D = L \quad D = R \]

\[ q, q' \in Q \setminus (Q_A \sqcup Q_R) \]

\[ q \in Q_A \sqcup Q_R \]
Building an Appropriate Structure (aka LEGO for Grownups)

The Robinson Tiling(s)
Canonical Robinson Tiling (Non-Overlapping Markers)

Figure 6: Hierarchical structure of the Robinson tiling.
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Figure 6: Hierarchical structure of the Robinson tiling.
Enhanced Robinson Tiling (Markers with Reconstruction)

Figure 7: A Robinson variant, with strengthened local rules.
Figure 8: Alternating Red-Black structure,
Two-Coloured Robinson for Turing Machines (Markers with Computation Area)

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Two-Coloured Robinson for Turing Machines (Markers with Computation Area)

Figure 8: Alternating Red-Black structure, with a sparse computation area.
Structural Properties of the Base Layer

- The $n$-macro-tile has a length $\ell_n = 2^n - 1$.
- The $n$-macro-tiles are non-overlapping.
- Any locally admissible window of length $2\ell_n + 5$ contains a $n$-macro-tile. (Gayral, Sablik, and Taati 2023, Lemma 29)
- The $N$-th Red square occurs in a $(2N + 1)$-macro-tile.
- The $N$-th Red square has a length $4^N + 1$.
- The $N$-th Red square has a sparse computing area of size $2^N + 1$. 
Building an Appropriate Structure (aka LEGO for Grownups)

Structure for Entropy Control
Hot and Frozen Areas

Red squares may be Blocking, with a Hot exterior and Frozen core. The rest must locally synchronise on Hot or Frozen.

Figure 9: Admissible configurations for Hot and Frozen squares.
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Figure 9: Admissible configurations for Hot and Frozen squares.
We (can) unary encode $N$ as an input for computations in the $N$-th Red square. We check whether $N = 3^k$. If not, the Red square cannot be Blocking.

**Figure 10:** The 2nd scale of Red squares cannot be Blocking.
Scales for the Marker Sets

• $Q_k$ the set of $n_k := (2 \times 3^k + 1)$ Robinson macro-tiles on the window $B_k := I_{\ell_{n_k}}$, the $3^k$-th scale of locally admissible tiles with Red squares.

• A $(k + 1)$-marker is a grid of $16^{3^k} \times 16^{3^k}$ smaller $k$-markers.

• This structure has positive entropy as each 0-marker, which occur periodically, can have a different state (either Hot or Blocking).
We implement an odometer in $k$-markers, that cycles with period $t_k = 2^\left\lfloor \log_2 \left( \left\lfloor \log_2 (k) \right\rfloor \right) \right\rfloor$, so that Red squares are Blocking once for each cycle.

Figure 11: The repartition of Frozen squares is forced by the odometer.

The Red square of a $(k + 1)$-marker initialises $k$-markers at 0 on one side.
Repartition of Frozen Tiles

Figure 12: Approximation of a Total Perspective Vortex.
(One 2-marker would be a $4096 \times 4096$ grid of such 1-markers.)
Density of Frozen Tiles

The average scale of Blocking squares in a $k$-marker goes to $\infty$ as $k \to \infty$.

**Proposition (Gayral, Sablik, and Taati 2023, Propositions 33 and 34)**

Fix a microscopic scale $m$.

The proportion of non-Frozen $m$-markers in a $k$-marker is of order:

$$
\prod_{j=m+1}^{k} \left( 1 - \frac{1}{4t_j} \right) \xrightarrow{k \to \infty} 0.
$$

Thus, generically, a globally admissible tiling is totally Frozen.

We are back to a uniquely ergodic zero-entropy case.

However, this rigid structure, with gaps in the scales, will allow us to slow down the speed of

$$
\frac{\log_2(\#Q_k)}{\#B_k} \to 0.
$$
Encode a letter on Red lines so that:

- Blocking and Hot squares are labelled 0,
- Frozen squares are labelled ±1,
- Neighbouring Frozen squares synchronise their bit.

A Blocking $k$-marker central square encodes a binary word of length $3^k - 1$.

Generically, a (Frozen) tiling encodes a sequence of bits in $\{±1\}^\mathbb{N}$.

Globally admissible tilings still have zero-entropy, but now we have a source of entropy for locally admissible markers.
Counting Markers

Let $Q_k = Q^H_k \sqcup Q^B_k \sqcup Q^F_k$ depending on whether the Red square is Hot, Blocking or Frozen.

**Proposition (Gayral, Sablik, and Taati 2023, Lemma 31, Propositions 42 and 43)**

We have:

- $\#Q^H_k \approx C_1^{16^k} \, 3^k$ with $2^4 - k \leq C_k \leq 2$,
- $\#Q^B_k \approx (\#Q^H_k)^{\frac{3}{4}}$,
- $\#Q^F_k \leq C_4^{3^k}$ for some $C > 1$.

Thus, $\#Q_k \approx \#Q^H_k$. 
We conclude that $\mu_{Q_k}$ is close to the uniform distribution on $Q^H_k$.

**Figure 13:** In the weak-$*$ topology, Gibbs measures are approximately grids of uniform markers.
A (Uniformly) Stable Structure

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**Figure 13:** In the weak-$\ast$ topology, Gibbs measures are approximately grids of uniform markers.

The induced model is uniform, stable, and the limit measure corresponds to $U(\{\pm1\}^\mathbb{N})$. 
Forcing Complex Structures
Forcing a Distribution on Words

We can embed a Turing machine on a new layer to simulate a non-uniform distribution on the word encoded in each Blocking square.

This will easily give us uniformly chaotic models, e.g. by simulating $\delta_0$, then $\delta_{11}$, $\delta_{000}$ and so on, so that $G_\sigma(\infty)$ corresponds to $[\delta_{0^n}, \delta_{1^n}]$.

What kind of kind of sets $G_\sigma(\infty)$ we can obtain for this class of uniform models?
Computational Complexity of Uncountable Sets

Let \((X, d)\) a metric space with a countable dense basis \(\mathcal{B}\).

**Definition**

Let \(Y \subset X\) be a closed set and \(\mathcal{N}(Y) := \{(x, r) \in \mathcal{B} \times \mathbb{Q}^+, \overline{B(x, r)} \cap Y \neq \emptyset\}\).

The set \(Y\) is said to be \(\Pi_k\)-computable *iff* the countable set \(\mathcal{N}(Y)\) is, *i.e.* there is a computable \(\varphi\) such that:

\[
(x, r) \in \mathcal{N}(Y) \iff \forall y_1, \exists y_2, \forall y_3, \ldots, \varphi(x, r, y_1, \ldots, y_k)
\]

Here, for invariant measures \(\mathcal{M}_\sigma(\Omega_A)\) with the weak-* topology, we use the periodic measures \(\hat{\delta}_w\), with \(w \in \mathcal{A}^{[0,n-1]}^d\), as a basis \(\mathcal{B}\).
Let $\varphi$ a computable potential, inducing a uniform model.

**Proposition (Gayral, Sablik, and Taati 2023, Proposition 3)**

There is a sequence $\beta_k \to \infty$ such that $\text{diam}(G_\sigma(\beta_k)) \to 0$ and $G_\sigma(\infty) = \text{Acc}(G_\sigma(\beta_k))$.

Without loss of generality, we have rational parameters $\beta_k \in \mathbb{Q}$.

**Theorem (Gayral, Sablik, and Taati 2023, Theorem 17)**

We have $B(x, r) \cap G_\sigma(\infty) \neq \emptyset$ iff:

$$\forall \varepsilon \in \mathbb{Q}^+, \forall \beta_0 \in \mathbb{Q}^+, \exists \beta \in \mathbb{Q}_{> \beta_0}^+, \exists y \in \mathcal{B}, G_\sigma(\beta) \subset B(y, \varepsilon) \text{ and } B(y, \varepsilon) \cap B(x, r) \neq \emptyset.$$ 

Consequently, we have a $\Pi_2$ upper bound on the complexity of $G_\sigma(\infty)$. 
Proposition (Gayral, Sablik, and Taati 2023, Proposition 5)

There is a characterisation of $\Pi_2$-computable sets through accumulation points:

\[ Y \in \Pi_2 \iff Y = \text{Acc} (x_n) \text{ with } (x_n) \in B^\mathbb{N} \text{ computable.} \]

\[ Y \in \Pi_2 \text{ and connected} \iff Y = \text{Acc} (x_n) \text{ with } (x_n) \in B^\mathbb{N} \text{ computable, and } d(x_n, x_{n+1}) \to 0. \]

Thus, we can embed the Turing machine computing any such sequence, to obtain any $\Pi_2$ connected subset of $\mathcal{M} (\{\pm 1\}^\mathbb{N})$ encoded in $G_\sigma(\infty)$. 


THE END OF PRESENTATION

ONE MORE SLIDE:

Thank you.