Uniformly Chaotic Finite-Range Lattice Models

And the Characterisation of the Set of Ground States Thereof

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Joint work with Mathieu Sablik (IMT) and Siamak Taati (AUB, Beirut)

Thermodynamic Formalism and Chaoticity

Gibbs Measures on Finite Spaces

- Ω a *finite* set of *states*.
- $E : \Omega \to \mathbb{R}^+$ an *energy* function.
- β the inverse temperature.

Theorem (Variational Principle)

The distribution $\mu_\beta(\omega) \propto \exp\left(-\beta E(\omega)\right)$ *is the only maximiser of* $\mu \mapsto H(\mu) - \beta\mu(E)$*, with* $H(\mu) := \sum -\log_2(\mu(\omega))\mu(\omega)$ *the entropy.*

We call μ_β *a Gibbs measure.*

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We call μ_β a Gibbs measure.

- At high temperatures, as $\beta \to 0$, we converge to the uniform distribution $\mathcal{U}(\Omega)$, that maximises H.
- At low temperatures, as $\beta \to \infty$, we converge to the uniform distribution $\mathcal{U}(\Omega^*)$, that maximises H among measures of minimal energy, supported by $\Omega^* := \arg\min(E)$.

Invariant Gibbs Measures on Lattice Models

- $\Omega_{\mathcal{A}} := \mathcal{A}^{\mathbb{Z}^d}$ the phase space, with $\mathcal{A}$ a finite alphabet.
- $\mathbb{Z}^d \overset{\sigma}{\curvearrowright} \Omega_{\mathcal{A}}$ the shift action, so that $\sigma^x(\omega)_y = \omega_{y-x}$ for any $x, y \in \mathbb{Z}^d$ and $\omega \in \Omega_{\mathcal{A}}$.
- $\mathcal{M}_\sigma(\Omega_{\mathcal{A}})$ the set of invariant measures on $\Omega_{\mathcal{A}}$, such that $\mu \circ \sigma^x = \mu$ for any $x \in \mathbb{Z}^d$.
- $\varphi : \Omega_{\mathcal{A}} \to \mathbb{R}^+$ a continuous potential, the contribution of $0 \in \mathbb{Z}^d$ to the energy.

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Definition (Pressure Function)

Define the pressure $p_\mu(\beta) := h(\mu) - \beta\mu(\varphi)$,
with $h(\mu) := \lim \frac{1}{n^d} H\left(\mu_{[\![0,n-1]\!]^d}\right)$ the entropy per site.

Let $\mathcal{G}_\sigma(\beta) := \arg\max_{\mu \in \mathcal{M}_\sigma} p_\mu(\beta)$ the set of Gibbs measures.

- φ has finite range if it is *locally constant*, if $\varphi(\omega)$ only depends on $\omega_{[\![-r,r]\!]^d}$.

Limit Behaviour for Ground States

- We call $(\mu_\beta \in \mathcal{G}_\sigma(\beta))_{\beta>0}$ a *cooling trajectory* of the model.
- Denote $\mathcal{G}_\sigma(\infty) := \mathsf{Acc}_{\beta\to\infty}\,\mathcal{G}_\sigma(\beta)$ the set of *ground states*, of accumulation points of all the cooling trajectories.
- $\mathcal{G}_\sigma(\infty)$ is a connected compact set (for the weak-$*$ topology).

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Lemma

Assume that $X := \left\{\omega \in \Omega_{\mathcal{A}}, \forall x \in \mathbb{Z}^d, \varphi \circ \sigma^x(\omega) = 0\right\} \neq \emptyset$.
Then $\mathcal{G}_\sigma(\infty) \subset \mathcal{M}_\sigma(X)$*, and the ground states have maximal entropy* h *in* $\mathcal{M}_\sigma(X)$.

- Measures that maximise h in $\mathcal{M}_\sigma(X)$ are not necessarily in $\mathcal{G}_\sigma(\infty)$.

What can we ask about $\mathcal{G}_\sigma(\infty)$?

Stability and Chaos

Definition (Stability)

A model is stable if all the cooling trajectories converge to the same limit.

Definition (Chaoticity)

A model is chaotic if there is no converging cooling trajectory.

Definition (Uniformity)

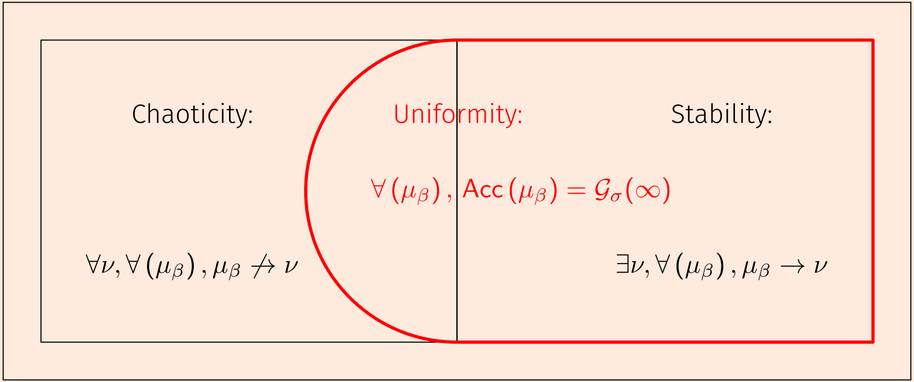
A model is uniform if all the cooling trajectories have the same accumulation set.

Recap of Behaviours

Chaoticity:	Stability:
$\forall \nu, \forall (\mu_\beta), \mu_\beta \not\to \nu$	$\exists \nu, \forall (\mu_\beta), \mu_\beta \to \nu$

Figure 1: Inventory and comparison of model behaviours.

Recap of Behaviours



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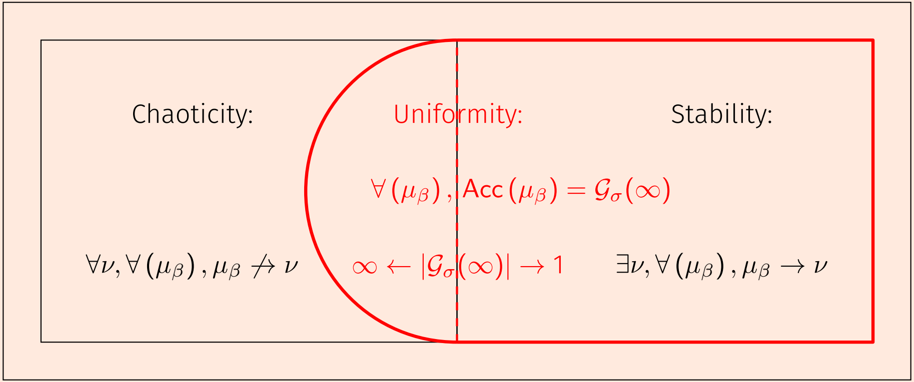

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Current Knowledge

Lemma

A one-dimensional finite range model induces a stable model.

Theorem (Chazottes and Hochman 2010)

There exists a one-dimensional potential inducing a chaotic model.

There exists a three-dimensional finite range potential inducing a chaotic model.

Theorem (Chazottes and Shinoda 2020; Barbieri et al. 2022)

There exists a two-dimensional finite range potential inducing a chaotic model.

Realisation Result on the Limit Set

- Remind that $\mathcal{G}_\sigma(\infty)$ must be connected.
- When φ is a *computable* potential inducing a uniform model, $\mathcal{G}_\sigma(\infty)$ must be a Π_2-computable set.

Theorem (Gayral, Sablik, and Taati 2023)

There exists a class of two-dimensional finite range potentials, inducing uniform models both stable and chaotic.

More precisely, we can realise any connected Π_2-computable compact set X as $\mathcal{G}_\sigma(\infty)$, up to a fixed computable affine homeomorphism.

Controlling Markers Distribution

General Idea for Chaoticity

We have two measures $\mu \neq \mu'$ s.t. $d(\mu, \mu') \geq r$ and:

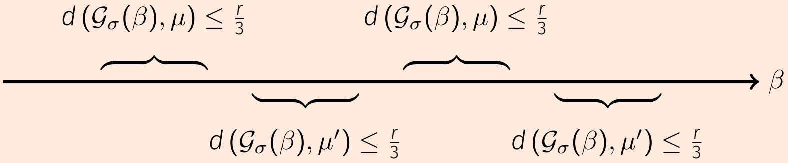

Figure 2: Alternating between mutually exclusive adherence values on non-overlapping intervals.

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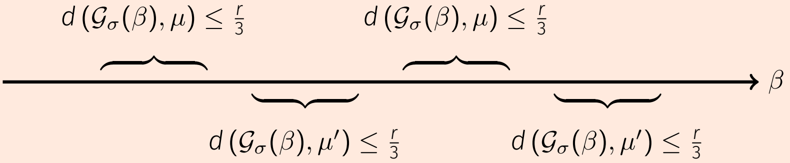

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Thus $\mathrm{Acc}(\mu_\beta)$ intersects the disjoint neighbourhoods of both μ and μ'.

General Idea for Uniformity

We want (μ_n) and $\varepsilon_n \to 0$ s.t.:

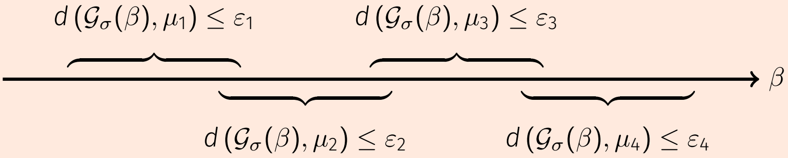

Figure 3: Contracting tube of measures with overlapping intervals.

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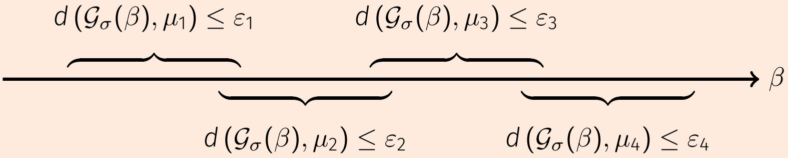

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Thus $\mathrm{Acc}\left(\mu_\beta\right) = \mathcal{G}_\sigma(\infty) = \mathrm{Acc}\left(\mu_n\right)$.

From Thermodynamics to Combinatorics

- $\mathcal{F}$ a finite set of *forbidden patterns* $w \in \mathcal{A}^{I(w)}$, each on a finite window $I(w) \Subset \mathbb{Z}^d$.
- $p \in \mathcal{A}^I$ is locally admissible if no translation of a forbidden pattern occurs within it.
- $\mathcal{F}$ induces a *subshift of finite type* (SFT) $X_\mathcal{F} \subset \Omega_\mathcal{A}$, closed and shift-invariant, made of configurations that are globally admissible.

Example

In one dimension, let $\mathcal{A} = \{\texttt{0}, \texttt{1}\}$ and $\mathcal{F} = \{\texttt{100}, \texttt{101}\}$. Then:

- $\texttt{0}^\mathbb{Z} \in X_\mathcal{F}$, $\texttt{1}^\mathbb{Z} \in X_\mathcal{F}$, $\cdots\texttt{000111}\cdots \in X_\mathcal{F}$,
- $\texttt{10}$ is locally admissible, but doesn't occur in any $\omega \in X_\mathcal{F}$.

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- 10 is locally admissible, but doesn't occur in any $\omega \in X_{\mathcal{F}}$.

Lemma

Assume that $X_{\mathcal{F}} \neq \emptyset$, and let $\varphi := \mathbb{1}_{\mathcal{F} \text{ covers } 0}$ the induced finite range potential. Then $\mathcal{G}_{\sigma}(\infty) \subset \mathcal{M}_{\sigma}(X_{\mathcal{F}})$, and the ground states have maximal entropy h in $\mathcal{M}_{\sigma}(X_{\mathcal{F}})$.

Control of Markers on a Temperature Interval

Definition (Marker Set with Margin Factor τ)

A *marker* set $Q \subset \mathcal{A}^{I_\ell}$ (with $I_\ell := [\![0, \ell-1]\!]^d$) is made of non-overlapping patterns, s.t. any locally admissible $\omega \in \mathcal{A}^{I_{(2+\tau)\ell-1}}$ must contain a marker somewhere.

Theorem (Adapted from Chazottes and Hochman 2010)

Denote G_n the locally admissible tilings of I_n, and μ_Q the cond. measure of μ on Q. We have constants C, C' s.t. for any marker set Q and $\varepsilon, \kappa > 0$, if

$$\frac{\log_2(\#G_n)}{\#I_n} \geq (1-\kappa)\frac{\log_2(\#Q)}{\#I_\ell} \quad \textit{and} \quad \beta \in \left[C\frac{\#I_\ell}{\varepsilon}, C'n\varepsilon\right]$$

then, for any $\mu \in \mathcal{G}_\sigma(\beta)$:

$$\mu(Q \text{ covers } 0) = 1 - O(\varepsilon+\tau) \quad \textit{and} \quad H(\mu_Q) \geq (1-2\kappa)\log_2(\#Q) - H(\kappa) - O(\varepsilon+\tau).$$

Building an Appropriate Structure (*aka* LEGO for Grownups)

Turing Machines as Tilings

Turing Machines

Turing machines are a model equivalent to real-life computers and algorithms.

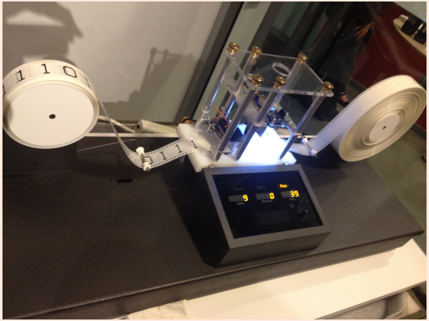

Figure 4: Real-life Turing machine (Source: wikimedia.org)

Formally, M is made of:

- internal states Q,
- an initial state $q_0 \in Q$,
- accepting states $Q_A \subset Q$,
- rejecting states $Q_R \subset Q$,
- an input alphabet $\mathcal{A}$,
- a tape alphabet $\Gamma \supset \mathcal{A} \sqcup \{\#\}$,
- a transition function $\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$.

Tileset of Space-Time Diagrams

A Turing machine $M = (Q, q_0, Q_A, Q_R, \mathcal{A}, \Gamma, \delta)$ can be simulated by a Wang tileset:

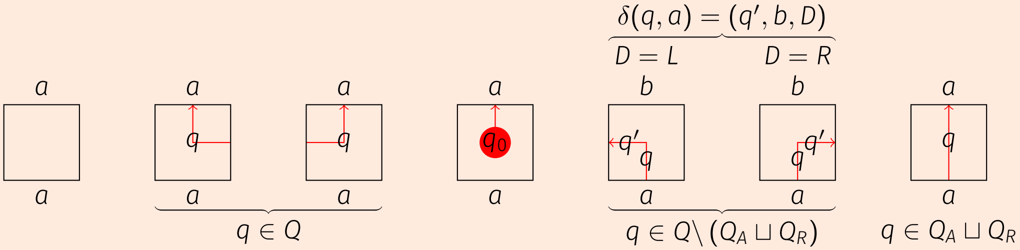

Figure 5: Turing space-time diagram Wang tiles for each letter $a \in \Gamma$.

Building an Appropriate Structure (*aka* LEGO for Grownups)

The Robinson Tiling(s)

Canonical Robinson Tiling (Non-Overlapping Markers)

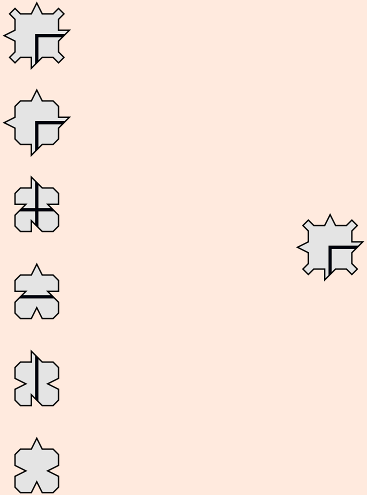

Figure 6: Hierarchical structure of the Robinson tiling.

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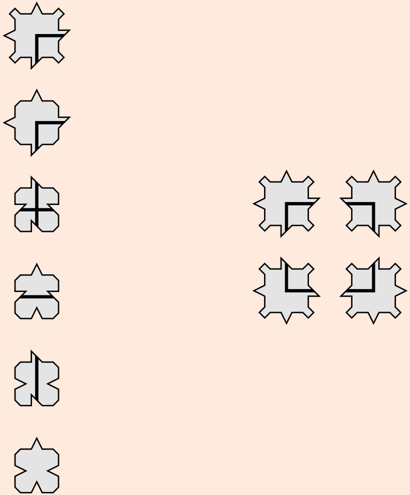

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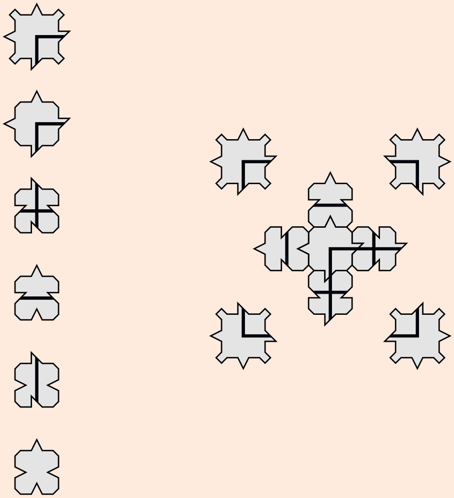

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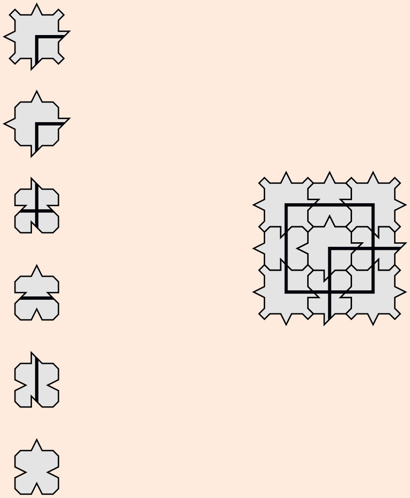

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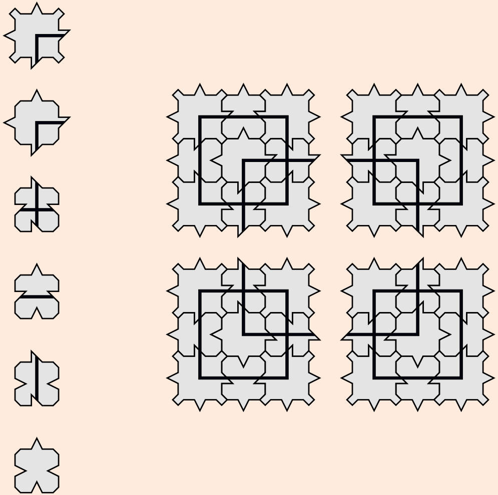

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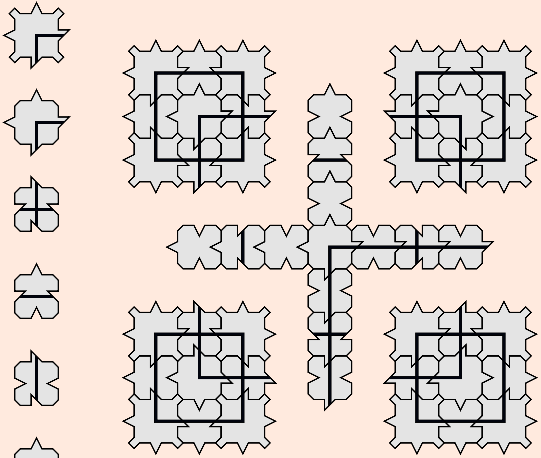

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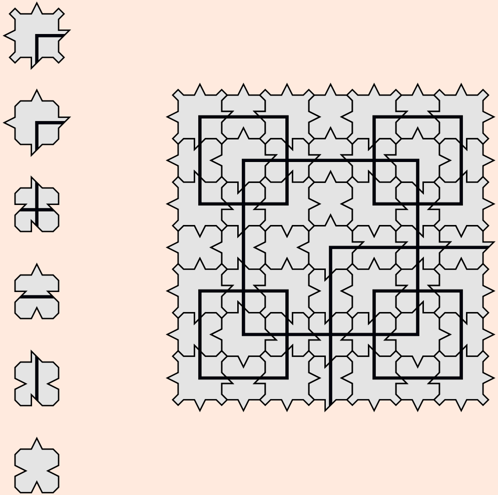

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Enhanced Robinson Tiling (Markers with Reconstruction)

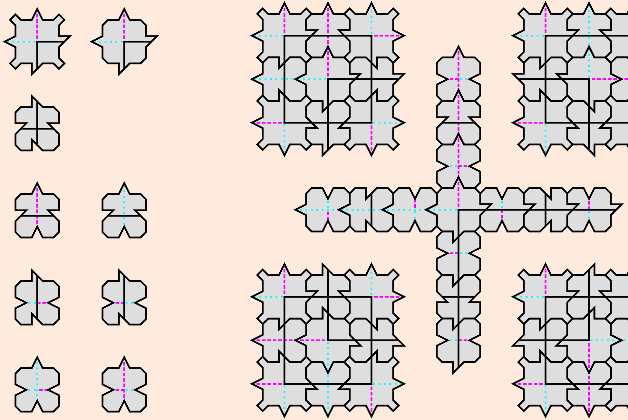


Figure 7: A Robinson variant, with strengthened local rules.

Two-Coloured Robinson for Turing Machines (Markers with Computation Area)

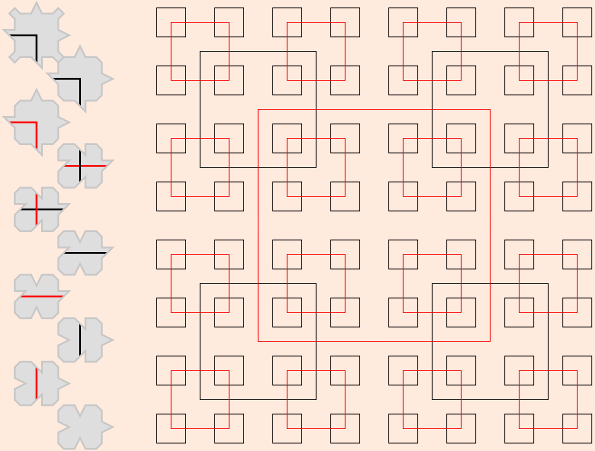
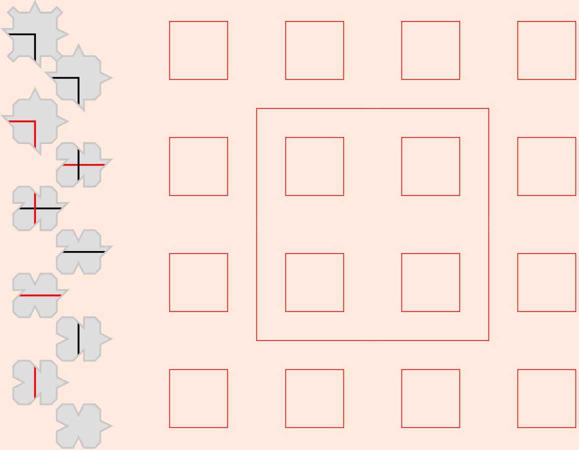


Figure 8: Alternating Red-Black structure,

Two-Coloured Robinson for Turing Machines (Markers with Computation Area)



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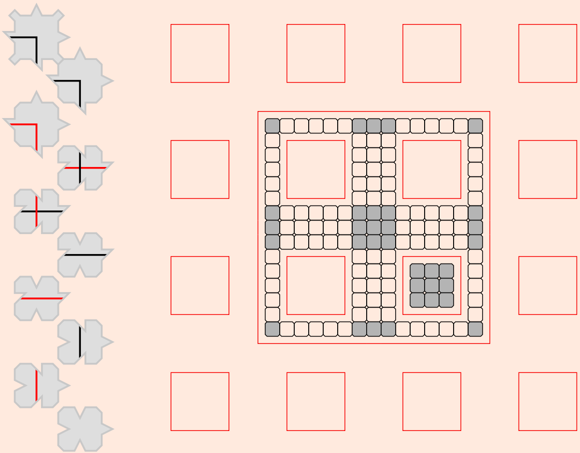

Figure 8: Alternating Red-Black structure, with a sparse computation area.

Structural Properties of the Base Layer

- The n-macro-tile has a length $\ell_n = 2^n - 1$.
- The n-macro-tiles are non-overlapping.
- Any locally admissible window of length $2\ell_n + 5$ contains a n-macro-tile. (Gayral, Sablik, and Taati 2023, Lemma 29)
- The N-th Red square occurs in a $(2N + 1)$-macro-tile.
- The N-th Red square has a length $4^N + 1$.
- The N-th Red square has a sparse computing area of size $2^N + 1$.

Building an Appropriate Structure (*aka* LEGO for Grownups)

Structure for Entropy Control

Hot and Frozen Areas

Red squares may be Blocking, with a Hot exterior and Frozen core.
The rest must locally synchronise on Hot or Frozen.

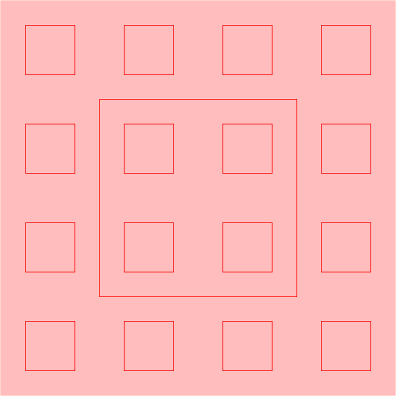

Figure 9: Admissible configurations for Hot and Frozen squares.

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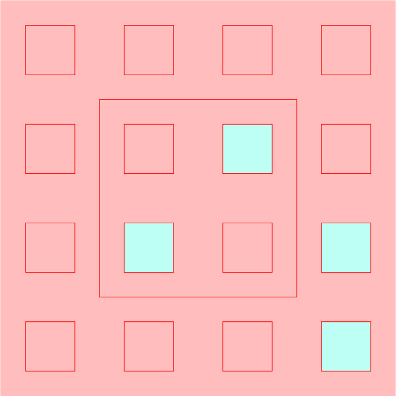

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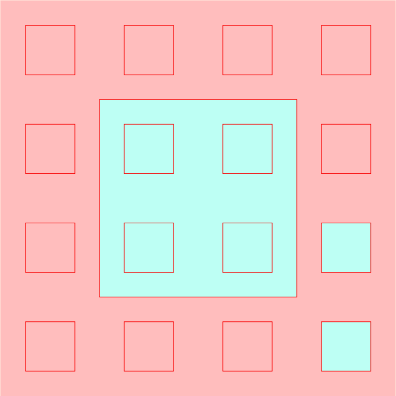

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Blockable Scales

We (can) unary encode N as an input for computations in the N-th Red square.
We check whether $N = 3^k$. If not, the Red square *cannot* be Blocking.

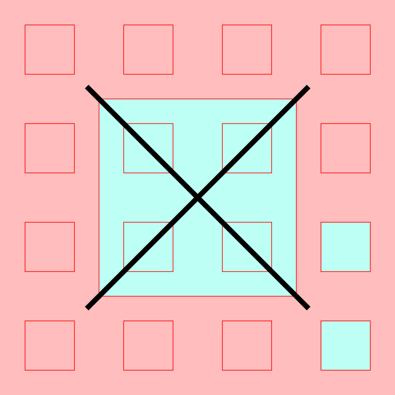

Figure 10: The 2nd scale of Red squares cannot be Blocking.

Scales for the Marker Sets

- Q_k the set of $n_k := \left(2 \times 3^k + 1\right)$ Robinson macro-tiles on the window $B_k := I_{\ell_{n_k}}$, the 3^k-th scale of locally admissible tiles with Red squares.

- A $(k+1)$-marker is a grid of $16^{3^k} \times 16^{3^k}$ smaller k-markers.

- This structure has positive entropy as each 0-marker, which occur periodically, can have a different state (either Hot or Blocking).

Odometer

We implement an odometer in k-markers, that cycles with period $t_k = 2^{\lfloor \log_2(\lfloor \log_2(k) \rfloor) \rfloor}$, so that Red squares are Blocking once for each cycle.

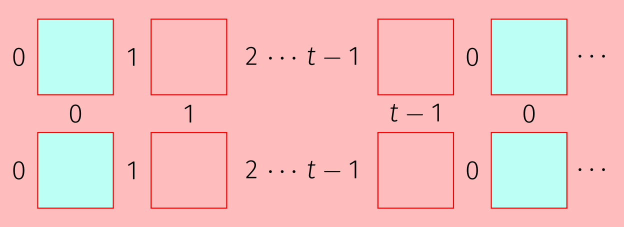

Figure 11: The repartition of Frozen squares is forced by the odometer.

The Red square of a $(k+1)$-marker initialises k-markers at 0 on one side.

Repartition of Frozen Tiles

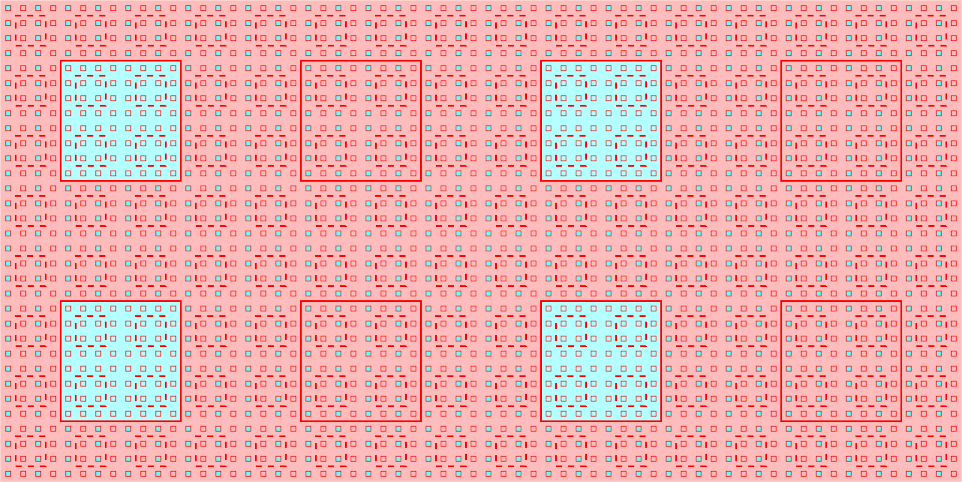

Figure 12: Approximation of a *Total Perspective Vortex*.
(One 2-marker would be a 4096 × 4096 grid of such 1-markers.)

Density of Frozen Tiles

The average scale of Blocking squares in a k-marker goes to ∞ as $k \to \infty$.

Proposition (Gayral, Sablik, and Taati 2023, Propositions 33 and 34)

Fix a microscopic scale m.
The proportion of non-Frozen m-markers in a k-marker is of order:

$$\prod_{j=m+1}^{k} \left(1 - \frac{1}{4t_j}\right) \xrightarrow[k\to\infty]{} 0.$$

Thus, generically, a globally admissible tiling is totally Frozen.

We are back to a uniquely ergodic zero-entropy case.

However, this rigid structure, with gaps in the scales, will allow us to slow down the speed of $\frac{\log_2(\#Q_k)}{\#B_k} \to 0$.

Words and Entropy

Encode a letter on Red lines so that:

- Blocking and Hot squares are labelled $\mathtt{0}$,
- Frozen squares are labelled $\mathtt{\pm 1}$,
- Neighbouring Frozen squares synchronise their bit.

A Blocking k-marker central square encodes a binary word of length $3^k - 1$.

Generically, a (Frozen) tiling encodes a sequence of bits in $\{\mathtt{\pm 1}\}^{\mathbb{N}}$.

Globally admissible tilings still have zero-entropy,
but now we have a source of entropy for locally admissible markers.

Counting Markers

Let $Q_k = Q_k^{\mathsf{H}} \sqcup Q_k^{\mathsf{B}} \sqcup Q_k^{\mathsf{F}}$ depending on whether the Red square is Hot, Blocking or Frozen.

Proposition (Gayral, Sablik, and Taati 2023, Lemma 31, Propositions 42 and 43)

We have:

- $\#Q_k^{\mathsf{H}} \approx C_k^{16^{3^k}}$ *with* $2^{4^{-k}} \leq C_k \leq 2$,
- $\#Q_k^{\mathsf{B}} \approx \left(\#Q_k^{H}\right)^{\frac{3}{4}}$,
- $\#Q_k^{\mathsf{F}} \leq C^{4^{3^k}}$ *for some* $C > 1$.

Thus, $\#Q_k \approx \#Q_k^{\mathsf{H}}$.

A (Uniformly) Stable Structure

We conclude that μ_{Q_k} is close to the uniform distribution on Q_k^{H}.

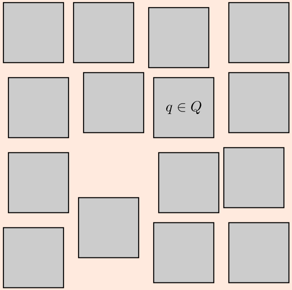

Figure 13: In the weak-$*$ topology, Gibbs measures are approximately grids of uniform markers.

A (Uniformly) Stable Structure

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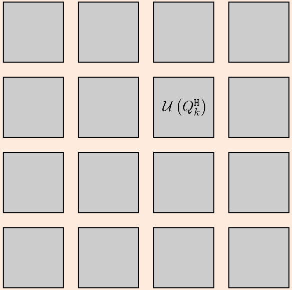

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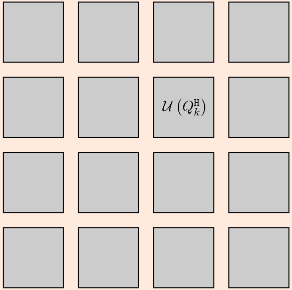

Figure 13: In the weak-$*$ topology, Gibbs measures are approximately grids of uniform markers.

The induced model is uniform, stable, and the limit measure corresponds to $\mathcal{U}\left(\{\pm 1\}^{\mathbb{N}}\right)$.

Forcing Complex Structures

Forcing a Distribution on Words

We can embed a Turing machine on a new layer
to simulate a *non-uniform* distribution on the word encoded in each Blocking square.

This will easily give us uniformly chaotic models,
e.g. by simulating $\delta_{\mathtt{0}}$, then $\delta_{\mathtt{11}}$, $\delta_{\mathtt{000}}$ and so on,
so that $\mathcal{G}_\sigma(\infty)$ corresponds to $[\delta_{\mathtt{0}^{\mathbb{N}}}, \delta_{\mathtt{1}^{\mathbb{N}}}]$.

What kind of kind of sets $\mathcal{G}_\sigma(\infty)$ we can obtain for this class of uniform models?

Computational Complexity of Uncountable Sets

Let (X, d) a metric space with a countable dense basis $\mathcal{B}$.

Definition

Let $Y \subset X$ be a closed set and $\mathcal{N}(Y) := \{(x, r) \in \mathcal{B} \times \mathbb{Q}^{+*}, \overline{B(x, r)} \cap Y \neq \emptyset\}$.

The set Y is said to be Π_k-computable *iff* the countable set $\mathcal{N}(Y)$ is,
i.e. there is a computable φ such that:

$$(x, r) \in \mathcal{N}(Y) \Leftrightarrow \forall y_1, \exists y_2, \forall y_3, \ldots, \varphi(x, r, y_1, \ldots, y_k)$$

Here, for invariant measures $\mathcal{M}_\sigma(\Omega_\mathcal{A})$ with the weak-* topology,
we use the periodic measures $\widehat{\delta_w}$, with $w \in \mathcal{A}^{[\![0,n-1]\!]^d}$, as a basis $\mathcal{B}$.

Uniform Upper Bound

Let φ a computable potential, inducing a uniform model.

Proposition (Gayral, Sablik, and Taati 2023, Proposition 3)

There is a sequence $\beta_k \to \infty$ such that $\operatorname{diam}(\mathcal{G}_\sigma(\beta_k)) \to 0$ and $\mathcal{G}_\sigma(\infty) = \operatorname{Acc}(\mathcal{G}_\sigma(\beta_k))$.

Without loss of generality, we have rational parameters $\beta_k \in \mathbb{Q}$.

Theorem (Gayral, Sablik, and Taati 2023, Theorem 17)

We have $\overline{B(x,r)} \cap \mathcal{G}_\sigma(\infty) \neq \emptyset$ iff:

$$\forall \varepsilon \in \mathbb{Q}^{+*}, \forall \beta_0 \in \mathbb{Q}^{+*}, \qquad \exists \beta \in \mathbb{Q}^{+*}_{>\beta_0}, \exists y \in \mathcal{B},$$
$$\mathcal{G}_\sigma(\beta) \subset B(y,\varepsilon) \text{ and } B(y,\varepsilon) \cap \overline{B(x,r)} \neq \emptyset.$$

Consequently, we have a Π_2 upper bound on the complexity of $\mathcal{G}_\sigma(\infty)$.

Equivalent Characterisation of Π_2

Proposition (Gayral, Sablik, and Taati 2023, Proposition 5)

There is a characterisation of Π_2-computable sets through accumulation points:

$$\begin{aligned} Y \in \Pi_2 \quad &\Leftrightarrow \quad Y = \mathrm{Acc}(x_n) \text{ with } (x_n) \in \mathcal{B}^{\mathbb{N}} \text{ computable.} \\ Y \in \Pi_2 \text{ and connected} \quad &\Leftrightarrow \quad Y = \mathrm{Acc}(x_n) \text{ with } (x_n) \in \mathcal{B}^{\mathbb{N}} \text{ computable,} \\ &\qquad \text{and } d(x_n, x_{n+1}) \to 0. \end{aligned}$$

Thus, we can embed the Turing machine computing any such sequence, to obtain any Π_2 connected subset of $\mathcal{M}\left(\{\pm 1\}^{\mathbb{N}}\right)$ encoded in $\mathcal{G}_\sigma(\infty)$.

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THE END OF PRESENTATION

ONE MORE SLIDE:

Thank you.