Trapspaces of Boolean networks

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Bringing memory to Boolean networks: a unifying framework
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Outline

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Boolean networks

A (Boolean) configuration is $x = (x_1, \ldots, x_n) \in \mathbb{B}^n$.

A Boolean network (BN) of dimension n is a mapping $f : \mathbb{B}^n \to \mathbb{B}^n$. It can be decomposed as

$$f = (f_1, \ldots, f_n), \quad \text{where } f_i : \mathbb{B}^n \to \mathbb{B} \ \forall i$$

We denote the set of BNs of dimension n as $\mathrm{F}(n)$.

For any $i \in [n]$ and any $f \in \mathrm{F}(n)$, the update of i according to f is represented by the BN $f^{(i)} \in \mathrm{F}(n)$ where

$$f^{(i)}(x) = (f_i(x), x_{-i}).$$

The asynchronous graph of $f \in \mathrm{F}(n)$ is $\mathtt{A}(f) = (V, E)$ where $V = \mathbb{B}^n$ and

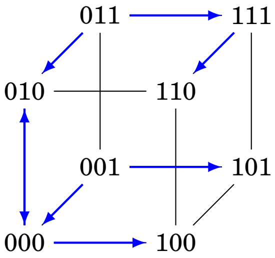
$$E = \left\{ (x, f^{(i)}(x)) : x \in \mathbb{B}^n, i \in [n] \right\}.$$

Example: asynchronous graph

Let $f \in \mathrm{F}(3)$ be defined as

x	$f(x)$
000	110
001	100
010	000
011	110
100	100
101	101
110	110
111	110

$\mathtt{A}(f)$: The asynchronous graph of f is given by



General asynchronous graph

For $f \in \mathrm{F}(n)$ and any $S \subseteq [n]$, the update of all the components in S is represented by $f^{(S)} \in \mathrm{F}(n)$ with

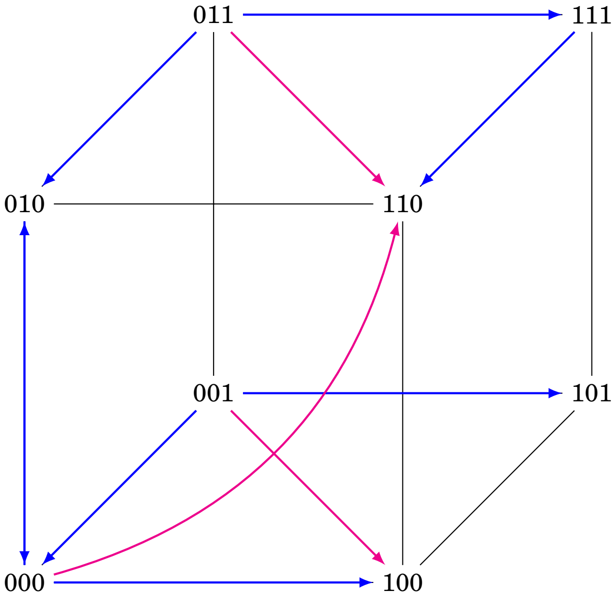
$$f^{(S)}(x) = (f_S(x), x_{-S}).$$

The general asynchronous graph of $f \in \mathrm{F}(n)$ is $\mathtt{GA}(f) = (V, E)$ where $V = \mathbb{B}^n$ and

$$E = \left\{(x, f^{(S)}(x)) : x \in \mathbb{B}^n, S \subseteq [n]\right\}.$$

Example: general asynchronous graph

A(f), GA(f): The general asynchronous graph of f is given by



Outline

Trapspaces

A subcube of $\mathbb{B}^n$ is any $X \subseteq \mathbb{B}^n$ of the form

$$X = \{x : x_S = 0, x_T = 1\}.$$

For any $x \in X$, there is a unique $y \in X$ furthest away from x, which we denote $y = X - x$.

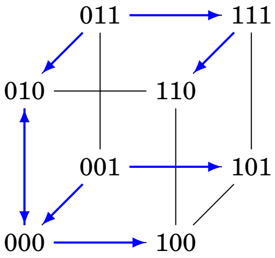
A trapspace of $f \in \mathrm{F}(n)$ is a subcube $X \subseteq \mathbb{B}^n$ such that the three equivalent conditions occur:

1. $f(X) \subseteq X$,
2. $f^{(i)}(x) \in X$ for all $i \in [n]$ and $x \in X$.
3. $f^{(S)}(x) \in X$ for all $S \subseteq [n]$ and $x \in X$.

The collection of all trapspaces of f is denoted by $\mathscr{T}(f)$.

The smallest trapspace of f that contains $x \in \mathbb{B}^n$ is the principal trapspace of x, which we denote by $T_f(x)$.
The collection of all principal trapspaces of f is denoted by $\mathscr{P}(f)$.

Example: trapspaces



We then have $\mathscr{T}(f) = \{A, B, C, D, E, F, G, H\}$, $\mathscr{P}(f) = \{A, B, C, D, E, F\}$ with

$$
\begin{aligned}
T_f(000) = T_f(010) = A &= \{x : x_3 = 0\} \\
T_f(111) = B &= \{x : x_{12} = 11\} \\
T_f(011) = T_f(001) = C &= \{x\} \\
T_f(110) = D &= \{x : x_{12} = 11, x_3 = 0\} \\
T_f(100) = E &= \{x : x_1 = 1, x_{23} = 00\} \\
T_f(101) = F &= \{x : x_{13} = 11, x_2 = 0\} \\
G &= \{x : x_1 = 1, x_3 = 0\} \\
H &= \{x : x_1 = 1\}.
\end{aligned}
$$

Trapping graph

The trapping graph of f is $\mathtt{T}(f) = (V, E)$ where $V = \mathbb{B}^n$ and

$$E = \{(x, y) : y \in T_f(x)\}.$$

The trapping closure of f is $f^{\mathrm{T}} \in \mathrm{F}(n)$, where

$$f^{\mathrm{T}}(x) = T_f(x) - x.$$

We then have $\mathtt{GA}(f^{\mathrm{T}}) = \mathtt{T}(f)$, which is transitive.

Proposition

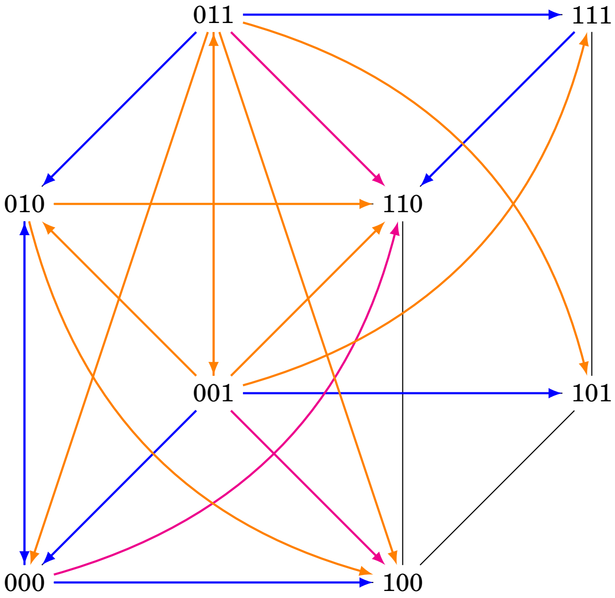
The following are equivalent for the BN g:

1. $g = f^{\mathrm{T}}$ *for some* f;
2. $g^{\mathrm{T}} = g$;
3. $\mathtt{GA}(g)$ *is transitive;*

in which case we say g is g is a trapping BN.

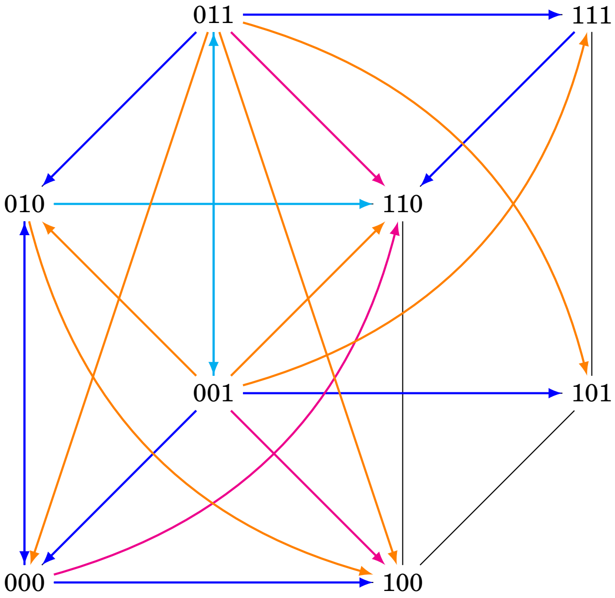
Example: trapping graph

A(f), GA(f), T(f): The trapping graph of f is given by



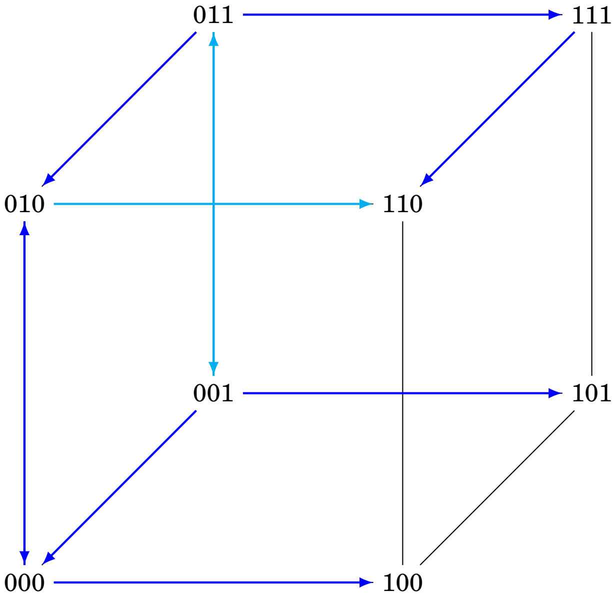
Example: trapping graph

$\mathtt{T}(f) = \mathtt{GA}(f^{\mathrm{T}})$: The trapping graph of f is given by



Example: trapping closure

$\mathtt{A}(f)$, $\mathtt{A}(f^{\mathrm{T}})$: The asynchronous graph of f^{T} is given by



Classification of trapping graphs

Say a graph $\Gamma = (V = \mathbb{B}^n, E)$ is pre-trapping if

1. Γ is transitive;
2. Γ is reflexive;
3. $N^{out}(x;\Gamma)$ is a subcube for all $x \in \mathbb{B}^n$.

For any pre-trapping $\Gamma = (V = \mathbb{B}^n, E)$, let $F(\Gamma) \in \mathrm{F}(n)$ such that

$$F(\Gamma)(x) = N^{out}(x;\Gamma) - x.$$

(In other words, $\mathtt{GA}(F(\Gamma)) = \Gamma$.)

Proposition

Γ is the trapping graph of a BN if and only if Γ is pre-trapping. For any pre-trapping graph Γ and any trapping network g, we have

$$\mathtt{GA}(F(\Gamma)) = \Gamma, \qquad F(\mathtt{GA}(g)) = g.$$

BNs with the same trapspaces

Theorem (G, Paulevé, Riva 24)

Let $f, g \in \mathrm{F}(n)$. The following are equivalent:

1. $\mathscr{P}(f) = \mathscr{P}(g)$;
   (same collection of principal trapspaces)
2. $\mathscr{T}(f) = \mathscr{T}(g)$;
   (same collection of trapspaces)
3. $T_f(x) = T_g(x)$ *for all* $x \in \mathbb{B}^n$;
   (same principal trapspace for each configuration)
4. $\mathtt{T}(f) = \mathtt{T}(g)$;
   (same trapping graph)
5. $f^{\mathrm{T}} = g^{\mathrm{T}}$.
   (same trapping closure)

Outline

Boolean networks

Trapspaces

Collections of trapspaces

Trapping and commutative networks

Classification of collections of principal trapspaces

Let $\mathscr{A}$ be a collection of subcubes of $\mathbb{B}^n$. For any $x \in \mathbb{B}^n$, denote the intersection of all the subcubes in $\mathscr{A}$ that contain x by

$$\mathscr{A}(x) = \bigcap \{A \in \mathscr{A} : x \in A\}.$$

So $T_f(x) = \mathscr{T}(f)(x) = \mathscr{P}(f)(x)$.

Let $\mathscr{Q}$ be a collection of subcubes of $\mathbb{B}^n$. We say $\mathscr{Q}$ is pre-principal if

$$\mathscr{Q} = \{\mathscr{Q}(x) : x \in \mathbb{B}^n\}.$$

Proposition

$\mathscr{Q}$ is pre-principal iff $\mathscr{Q} = \mathscr{P}(f)$ for some BN f.

Question

What is the complexity of recognising pre-principal collections of subcubes?

Classification of collections of trapspaces

We say a collection $\mathscr{J}$ of subcubes is pre-ideal if

1. $\mathbb{B}^n \in \mathscr{J}$;
2. if $A, B \in \mathscr{J}$ and $A \cap B \neq \emptyset$, then $A \cap B \in \mathscr{J}$;
3. for any subcollection $\mathscr{R} \subseteq \mathscr{J}$, if $R = \bigcup \mathscr{R}$ is a subcube, then $R \in \mathscr{J}$.

Proposition

$\mathscr{J}$ is pre-ideal iff $\mathscr{J} = \mathscr{T}(f)$ for some BN f.

Question

What is the complexity of recognising pre-ideal collections of subcubes?

Refinement

Let $\mathscr{A}$ be a collection of subcubes of $\mathbb{B}^n$, then define $F(\mathscr{A}) \in \mathrm{F}(n)$ by

$$F(\mathscr{A})(x) = \mathscr{A}(x) - x.$$

So $f^{\mathrm{T}} = F(\mathscr{T}(f)) = F(\mathscr{P}(f))$.

Theorem (G, Paulevé, Riva 24)

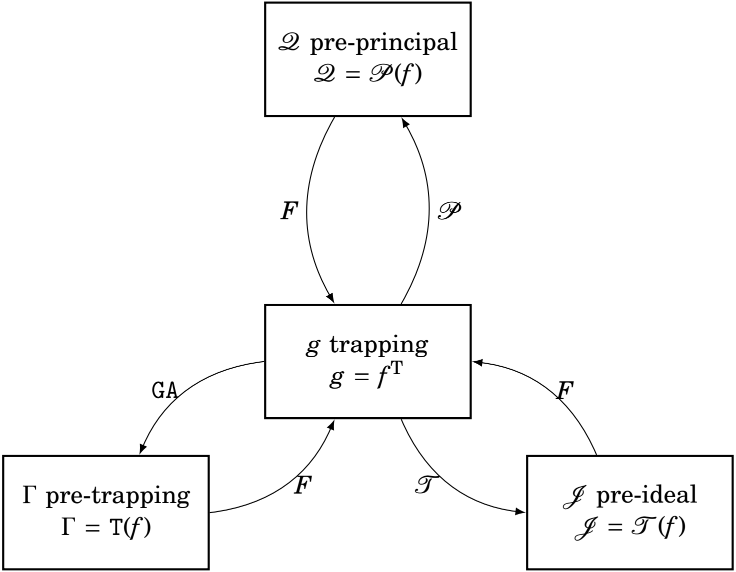
For any pre-principal collection $\mathscr{Q}$ and any trapping network g, we have

$$\mathscr{P}(F(\mathscr{Q})) = \mathscr{Q}, \qquad F(\mathscr{P}(g)) = g.$$

For any pre-ideal collection $\mathscr{J}$ and any trapping network g, we have

$$\mathscr{T}(F(\mathscr{J})) = \mathscr{J}, \qquad F(\mathscr{T}(g)) = g.$$

Summary


$\mathscr{Q}$ pre-principal
$\mathscr{Q} = \mathscr{P}(f)$
F
$\mathscr{P}$
g trapping
$g = f^{\mathrm{T}}$
GA
F
Γ pre-trapping
$\Gamma = \mathtt{T}(f)$
F
$\mathscr{T}$
$\mathscr{J}$ pre-ideal
$\mathscr{J} = \mathscr{T}(f)$

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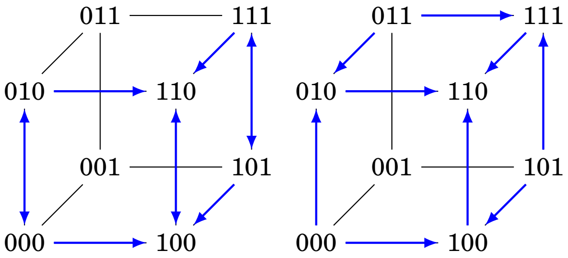
Trapping and commutative networks

Commutative networks

(Bridoux, G, Theyssier 20) A BN $f \in F(n)$ is commutative if for all $i, j \in [n]$, $f^{(i)}$ and $f^{(j)}$ commute.



Theorem ((G, Paulevé, Riva 24))

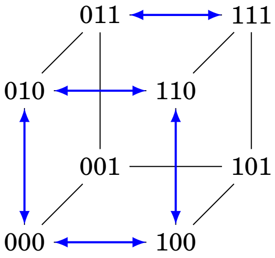
Commutative BNs are trapping.

Commutative networks are highly structured, and some of that structure generalises to trapping networks, e.g. they all have period at most two.

Negations on subcubes

A negation on subcubes is any $f = F(\mathscr{A})$, where $\mathscr{A}$ is a partition of $\mathbb{B}^n$ into subcubes.

Example with $\mathscr{A} = \{\{x_3 = 0\}, \{x_2x_3 = 11\}, \{x = 001\}, \{x = 101\}\}$:



A negation on subcubes is trapping because (pick one):

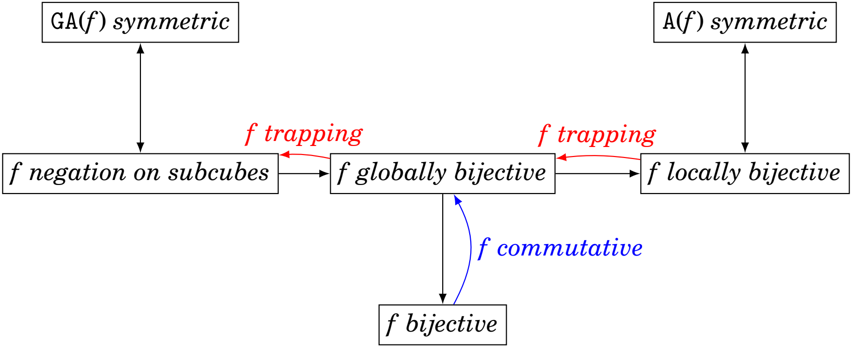
1. $\mathscr{A}$ is pre-ideal;
2. $\mathscr{A}$ is pre-principal;
3. f is commutative.

Beyond negations on subcubes

A BN f is

1. locally bijective if $f^{(i)}$ is bijective for all $i \in [n]$;
2. globally bijective if $f^{(S)}$ is bijective for all $S \subseteq [n]$.

Theorem ((Bridoux, G, Theyssier 20), (G, Paulevé, Riva 24))



Conclusion

Contributions:

- Trapping graph;
- Trapping vs commutative networks;
- Classification of (principal) trapspaces.

Outlook:

- Complexity of recognising collections of (principal) trapspaces.
- Classification of subclasses of trapping networks;
- Properties of negations on subcubes.