A Framework for Universality in Physics, Computer Science, and Beyond

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with Sebastian Stengele, Tobias Reinhart, Gemma De les Coves

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Motivation

Motivating examples

Primary:

- universal Turing machines
- universal spin models [De16]

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- universal Turing machines
- universal spin models [De16]

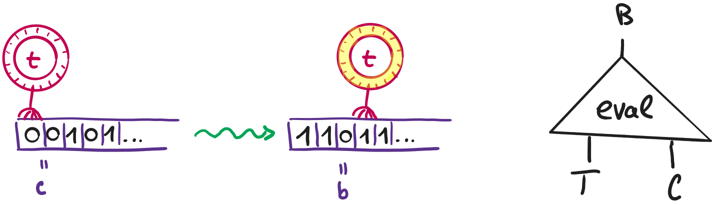
Others:

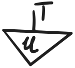
- NP-completeness
- universal neural networks
- generating sets (universal gate set, . . .)
- universal graphs
- universal grammar
- universal synthesis (e.g. chemical)
- universal morphogenesis in complex (e.g. biological) systems

Goals of the framework

- Unified language
- Examples of universality
- Knowledge-organization
- General theory of universality
  - necessary conditions for universality
  - Fixed-point theorem + relation to undecidability
  - trivial vs. non-trivial universality

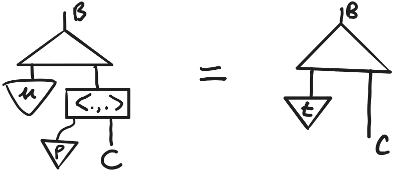
Universal Turing machine



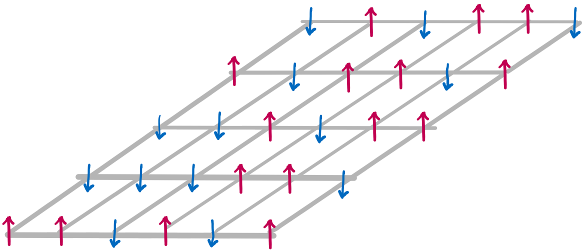


is universal ≈ u can emulate any $t \in T$:

$\exists$ p = 0 0 1 1 1 :



Classical spin systems



Models of

- magnetic materials, spin glasses
- phase transitions, criticality, percolation
- neural networks
- binding mechanisms in Biology, e.g. protein folding
- optimization problems in network theory

Spin system

- Spin d.o.f. Σ for each vertex in V
- A hypergraph (edges $\leftrightarrow$ local interactions)

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- Hamiltonian $\Sigma^V \to \mathbb{R}$ as a sum of local coupling terms

Spin system

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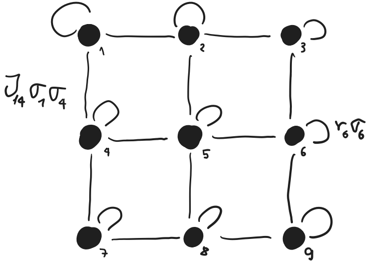
- ▷ A hypergraph (edges $\leftrightarrow$ local interactions)
- ▷ Hamiltonian $\Sigma^V \to \mathbb{R}$ as a sum of local coupling terms

2D Ising spin model with fields has $\Sigma = \mathbb{Z}_2$ and interaction lattice:

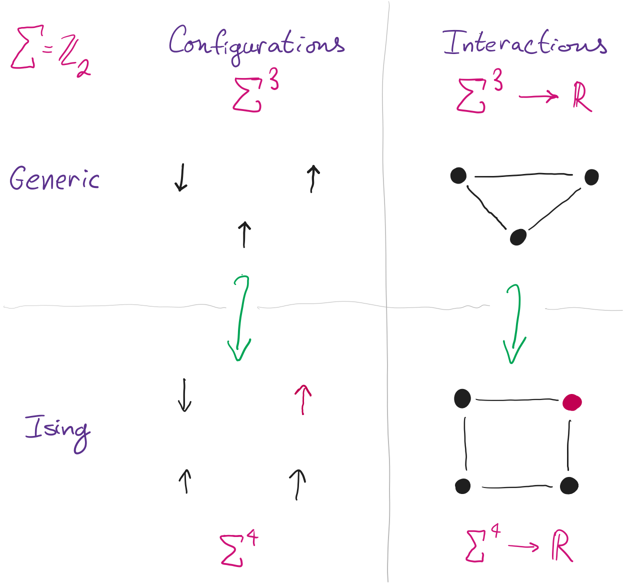


$$\sigma_i \in \Sigma$$

$$H = \sum_{i,j} J_{ij}\sigma_i\sigma_j + \sum_i r_i \sigma_i$$

Spin system simulation

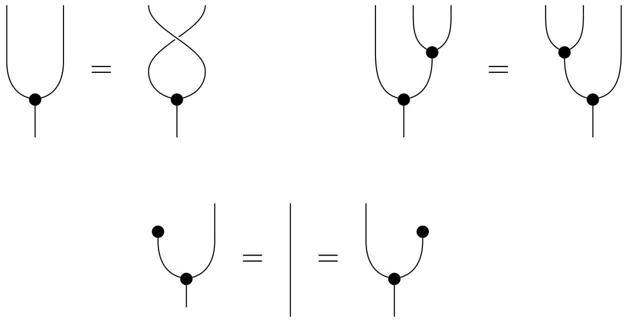
Every spin system can be simulated on a 2D Ising one [De16].



The set-up (simulators)

Ambient category

$\mathcal{A}$ is a **CD category**, an SMC with $X \to X \otimes X$ and $X \to I$ s.t.



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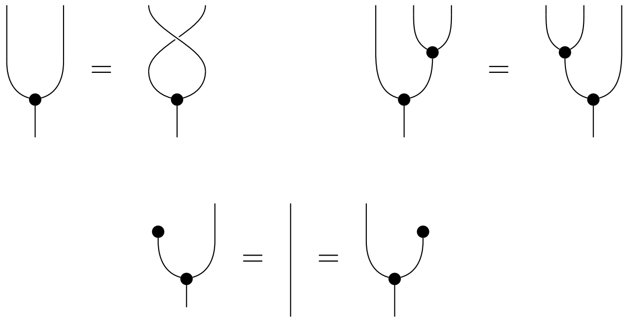
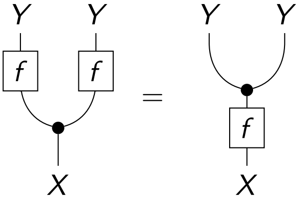


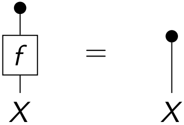
Key examples: Rel, Set, $\mathrm{Rel}_{\mathrm{poly}}$, Comp($\mathbb{N}$), Poly($\mathbb{N}$)

Deterministic

functional morphism:



total morphism:



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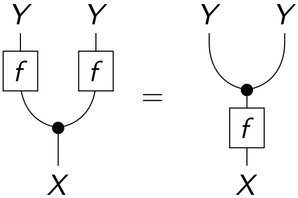

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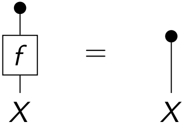
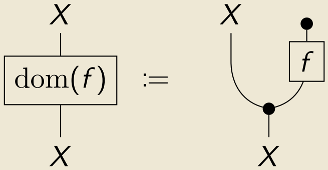


Functional + total = **deterministic**

Domain

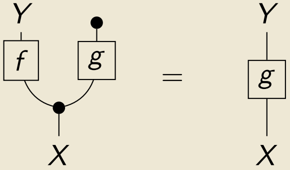
Definition ([Fr22])

The **domain** of $f\colon X \to Y$ is



Definition

f extends g, $f \sqsupseteq g$, if



Target–context category

Definition

A **target–context category**:

- a CD category $\mathcal{A}$; two objects $T, C \in \mathcal{A}$
- preorders $\gtrdot$ on every $\mathcal{A}(X, T \otimes C)$

such that $\forall\, f, g, h$:

- $f \sqsupseteq f$
- $f \sqsupseteq g \quad \Longrightarrow \quad f \gtrdot g$
- $f \gtrdot g \quad \Longrightarrow \quad f \circ h \gtrdot g \circ h$

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$\mathcal{A}$	**ambient cat.**	Comp($\mathbb{N}$) [Co08]
X	**object**	$\mathbb{N}^n$ for $n \in \mathbb{N}$
f	**morphism**	computable fun.
T	**targets**	Turing machines
C	**contexts**	input strings
$\gtrdot$	**ambient rel.**	computation

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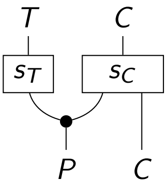
$\mathcal{A}$	**ambient cat.**	Comp($\mathbb{N}$) [Co08]	$\mathrm{Rel}_{\mathrm{poly}}$
X	**object**	$\mathbb{N}^n$ for $n \in \mathbb{N}$	"sized" sets
f	**morphism**	computable fun.	bounded relations
T	**targets**	Turing machines	spin systems
C	**contexts**	input strings	spin configurations
$\gtrdot$	**ambient rel.**	computation	energy condition

A simulator



$P \in \mathcal{A}$	**programs**
$s_T \colon P \to T$	**compiler**
$s_C \colon P \otimes C \to C$	**context reduction**

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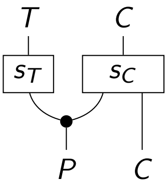

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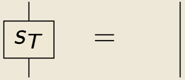

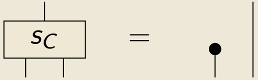

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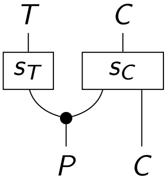

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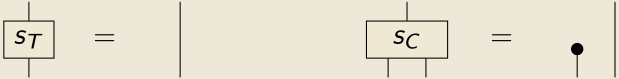
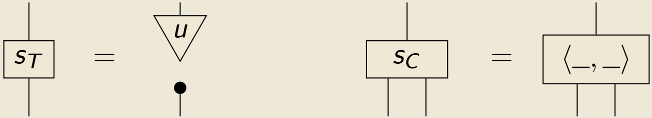


Example (singleton simulator for TM)

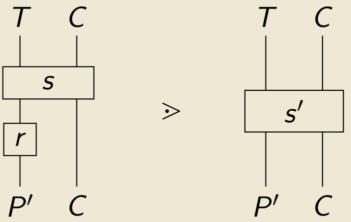


Universality

Reductions and universality

Definition

A **lax reduction** $r^* \colon s \to s'$ from simulator s to s' is a functional $r \colon P' \to P$ such that



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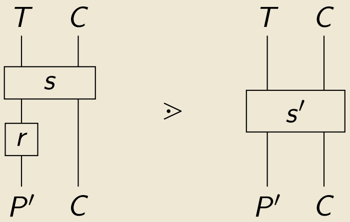

Definition

Simulator s is **universal** if there is a lax reduction $s \to \mathrm{id}$ to the trivial simulator.

Examples (universal simulators)

- Trivial simulator
- Singleton simulator for a universal TM

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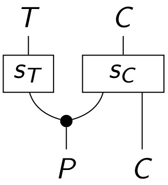
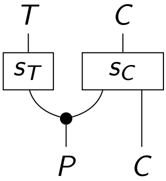
- A generating set ($T =$ tuples, $C =$ formulas)
- Universal Borel set
- Cofinal subset P of a poset $(T, \gtrdot)$.

No-go theorem



T	spin systems
$s_T\colon P \hookrightarrow T$	2D Ising
$s_C\colon P \otimes C \to C$	config. embedding

No-go theorem



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Theorem

For a "suitably $\succcurlyeq$-monotone" function $\varphi \colon T \to \mathbb{R}$, we have

$$s \text{ is universal} \quad \Longrightarrow \quad \sup \varphi\big(\mathrm{im}(s_T)\big) \geq \sup \varphi(T).$$

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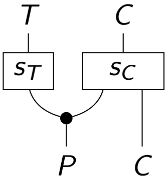

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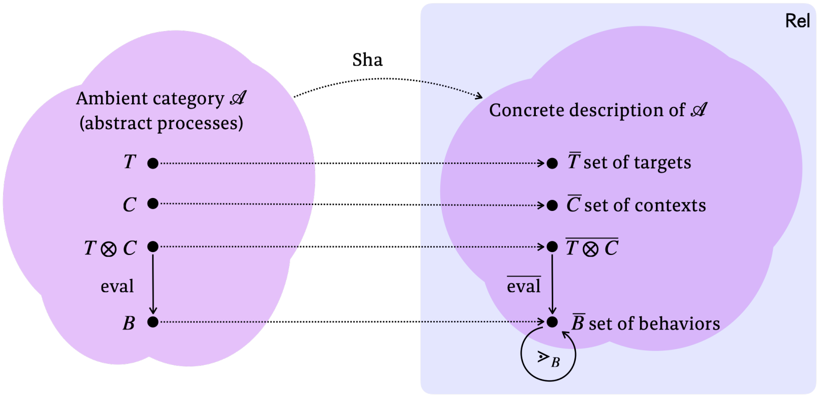
For spin systems, $\varphi = |\mathrm{spec}|$ works, and RHS is ∞, so

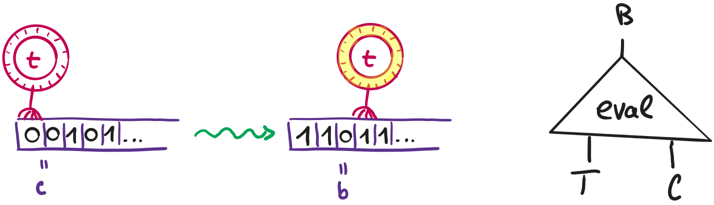
a universal spin model cannot be finite.

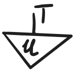
Relation to undecidability

Intrinsic behavior structure



Universal Turing machine





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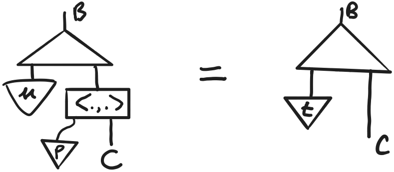

Intrinsic behavior structure

T	**targets**	TMs	spin systems
C	**contexts**	inputs	spin configurations
B	**behaviors**	outputs	energies $+ \ldots$
$\gtrdot_B$	preorder	$=$	$\ldots$
eval	$T \otimes C \to B$	evaluation	measurement

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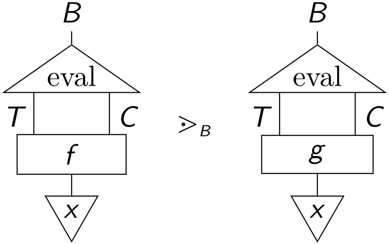
$f \gtrdot g$:



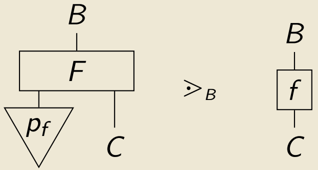
for all $x \in \mathcal{A}_{\mathrm{det}}(I, X)$, such that RHS is defined.

Fixed point theorem

Definition

$F\colon P \otimes C \to B$ is a **complete parametrization** (CP) if for every f

$$\exists\, p_f \in \mathcal{A}_{\mathrm{det}}(I, P) :$$



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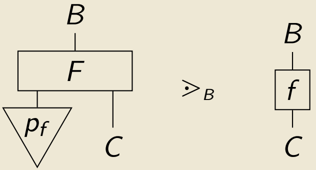
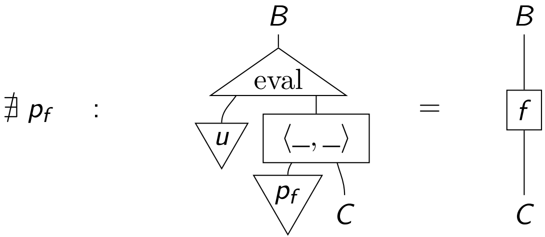


Theorem (Fixed Point Theorem à la [La69])

If $F\colon C \times C \to B$ is a CP, then every $g\colon B \to B$ has a (quasi) fixed point.

Unreachability from universality

A simulator s has **unreachability** if $\mathrm{eval} \circ s$ is **not** a CP.



Lemma

If eval *is a CP and* s *is universal, then* $\mathrm{eval} \circ s$ *is a CP.*

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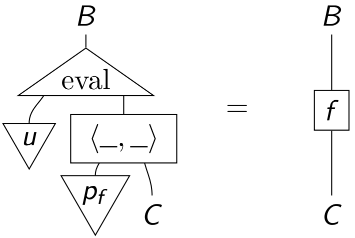

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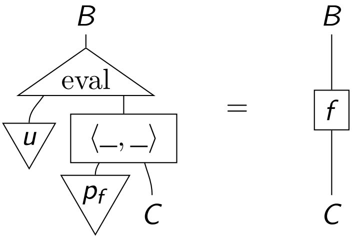

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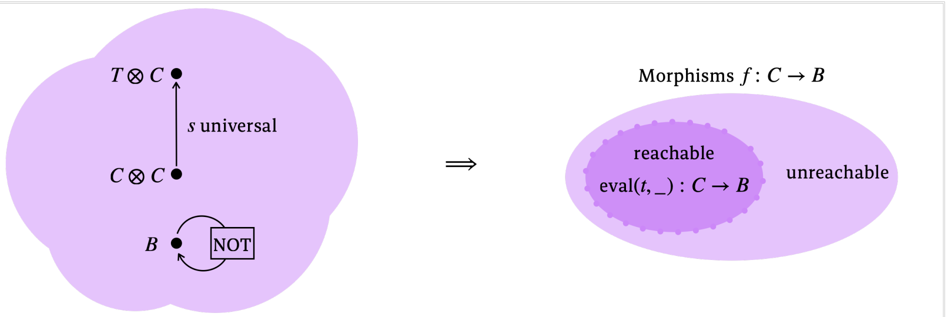
$$\text{universal } s + \text{fixed-point-free } g \overset{Lemma}{\Longrightarrow} \mathrm{eval} \text{ is not a CP}$$

$$\Longleftrightarrow \text{ unreachability of id}$$

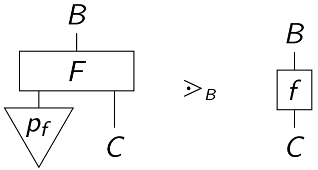
Unreachability from universality



Total fixed points

$F\colon P \otimes C \to B$ is a **complete parametrization** by total morphisms if for every f

$\exists\, p_f \in \mathcal{A}_{\mathrm{det}}(I, P)$:



Theorem (Total Fixed Point Theorem)

If $F\colon C \times C \to B$ **total** *and a CP* **by total morphisms**, *then every* $g\colon B \to B$ *has a* **total** *quasi-fixed point.*

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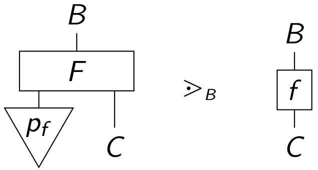

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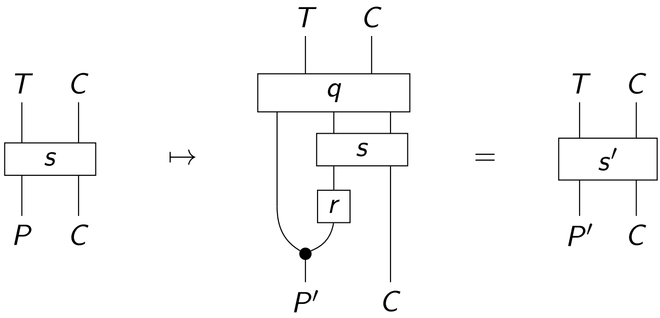
Corollary

There is no universal Turing machine that halts on every input.

Hierarchy of universal simulators

Simulator morphisms



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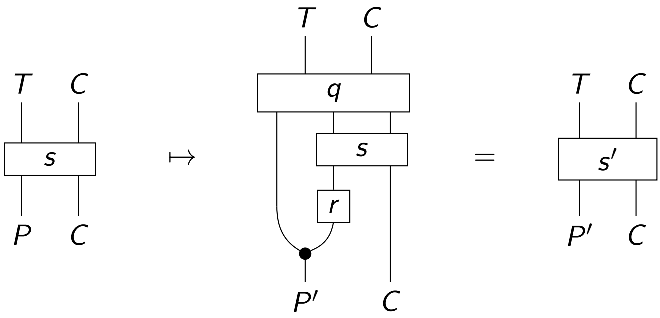
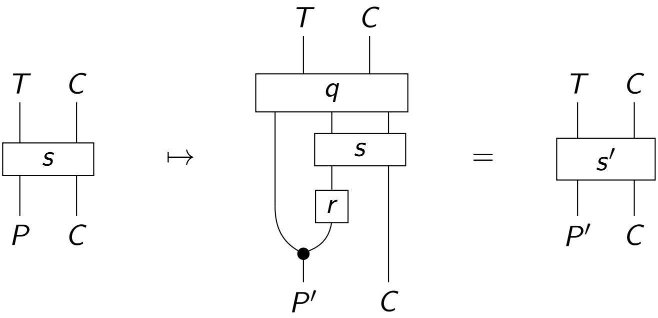


r is functional ensures sequential composition

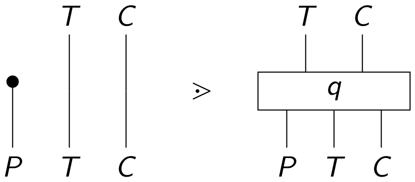
Simulator morphisms

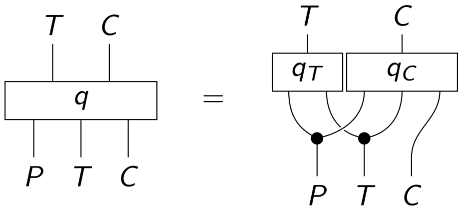


r is functional ensures sequential composition

We also require that (s' is universal) $\implies$ (s is universal).

Processings





Parsimony of simulators

Definition

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The singleton simulator s_u for a universal TM is strictly more parsimonious than the trivial simulator.

- $s_u \geq \mathrm{id}$ constructs right-inverse to the reduction.
- $s_u \not\leq \mathrm{id}$ because $\exists\, t, t' \colon I \to T$ such that
  - t cannot simulate t' and
  - they both compile to $u \in T$

Summary

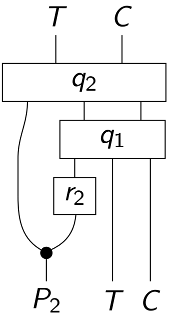
- Abstract notion of universality with several instances
- Necessary conditions for universality
- Fixed Point Theorem and unreachability
- Morphisms of simulators $\to$ non-trivial universality
- target–context functors & simulator categories

Morphism composition

Given two morphisms $(r_1^*, q_{1*})\colon s \to s_1$ and $(r_2^*, q_{2*})\colon s_1 \to s_2$ of simulators, we define the **sequential composition**

$$(r_2^*, q_{2*}) \circ (r_1^*, q_{1*})\colon s \to s_2$$

to be the morphism whose processing is given by the map



with reduction given by $(r_1 \circ r_2)^*$.

Cantor's Theorem

- C is an arbitrary set
- $T = 2^C$, the power set
- $B = \{0, 1\}$, $\gtrdot_B$ is equality
- $\mathrm{eval}(t, c) = t(c)$, the membership check
- $\mathrm{eval}\colon 2^C \times C \to 2$ is a complete parametrization of $C \to 2$.
- A universal simulator exists $\iff$ A surjection $C \to 2^C$ exists
- By the fixed point thm, there is no universal simulator.

Turing categories

Definition ([Co08])

A **Turing category** is a cartesian restriction category with a distinguished Turing object T and morphisms $\tau_{X,Y}\colon T \times X \to Y$ for any pair of objects X, Y such that for any $f\colon Z \times X \to Y$ there exists a unique $h\colon Z \to T$ satisfying $\tau \circ (h \times \mathrm{id}_X) = f$.

Example (Simulators of Turing machines)

Σ is a finite alphabet and $\Sigma^* = \cup_{n \geq 0} \Sigma^n$ its Kleene star. T is given by the set of Turing machines. Further objects are $C = \Sigma^* = B$ and finite products thereof. Morphisms are partial computable maps. There is a pairing function $\langle \_, \_ \rangle\colon C \times C \to C$.
The relation $\succcurlyeq_B$ is equality among strings and eval is given by $\tau_{C,C}$.

Ambient relations for TMs

Consider two TMs $t_1, t_2 \in T$.

In the target–context category Tur

$$t_1 \times \mathrm{id}_C \gtrdot t_2 \times \mathrm{id}_C \iff t_i \text{ compute the same partial function.}$$

t_1 halts $\implies$ t_2 halts.

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In the target–context category $\mathrm{Tur}^{\mathrm{int}}$

$$t_1 \times \mathrm{id}_C \gtrdot t_2 \times \mathrm{id}_C \iff t_1 \text{ computes an extension of } t_2.$$

t_1 halts $\not\Longrightarrow$ t_2 halts.

Polynomially bounded relations

Spin models

- $T = P =$ set of spin systems specified by size, interaction hypergraph, couplings, and fields.
- $C =$ spin configurations in Σ^*
- eval maps (t, c) to the energy in B.
- s_T takes a generic spin system t to a (larger) Ising system.
- s_C encodes its configurations into those of $s_T(t)$ via flag spins.

Spin models

Dense subset

- $T = \mathbb{R} \times \mathbb{R}_+$, i.e. points and precisions
- $C = I$, the singleton set
- $B = \mathcal{P}(\mathbb{R})$ with $\gtrdot_B$ the subset inclusion
- eval maps (t, δ) to the open ball of radius δ centered at t.
- $P = \mathbb{Q} \times \mathbb{R}_+$ and s_T is the inclusion into T
- The reduction $r \colon T \to P$ maps (t, δ) to $\left(q_{(t,\delta)}, \delta/2\right)$

Generating family (of a group)

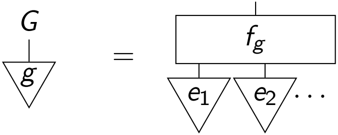
- $T = P = G^*$ are families of group elements
- C consists of formulas $G^k \to G$, e.g.

$$(g, h) \mapsto hg^{-1}h^2$$

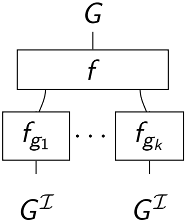
- $B = G$ with $\gtrdot_B$ the equality
- eval evaluates formulas on families with enough elements.

Generating family (of a group)

- s_T discards P and returns the generating family $(e_i)_{i \in \mathcal{I}}$
- For each g, we have a formula $f_g \colon G^{\mathcal{I}} \to G$ with



- s_C acts by mapping the pair of a family (g_j) and a formula $f \colon G^k \to G$ to

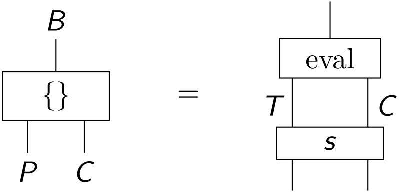


Weak limits

- $T = B =$ the set of cones over a given diagram $F\colon \mathsf{J} \to \mathsf{C}$
- $C = I$, the singleton set
- $\mathrm{eval} = \mathrm{id}_T$
- $\psi \gtrdot_B \phi$ if ϕ factors through ψ.
- $P = I$ and s_T is the weak limit of F.
- Can be generalized to scenarios when $\lim F$ doesn't exist by using other P.

Monoidal computer

Specify a family of universal evaluators [Pa18]



for fixed P, every $C \in \mathcal{A}$, and a corresponding w.p.s. eval and a universal simulator s.

Plus there are (deterministic) partial evaluators relating them: