Complete Equational Theories for Quantum circuits

Alexandre Clément

LMF, Inria QuaCS, Université Paris-Saclay

Joint work with Noé Delorme, Nicolas Heurtel, Shane Mansfield, Simon Perdrix, Benoît Valiron and Renaud Vilmart

Marseille, November 26, 2024

Introduction

Equational theory = a set of equations, aka non-oriented rewrite rules. E.g.

For example:

Can be useful for:

- Optimization
- Hardware constraint satisfaction
- Verification via equivalence checking
- Error correction, ...

Complete if any two equivalent circuits can be transformed into each other.

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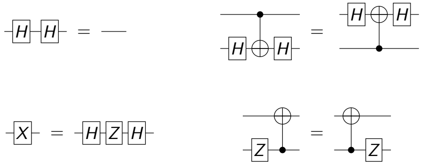

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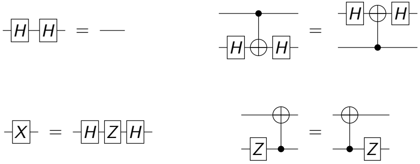

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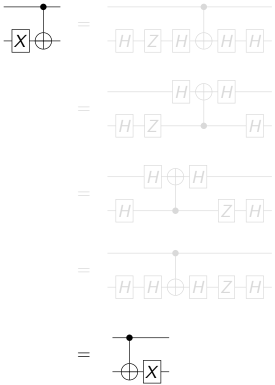

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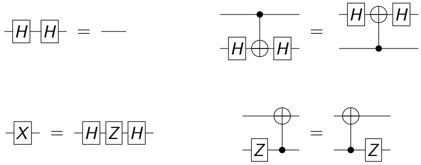

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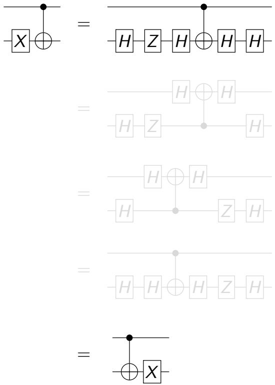

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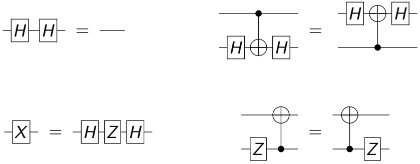

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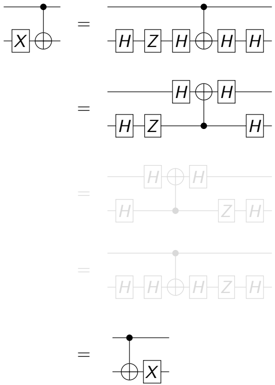

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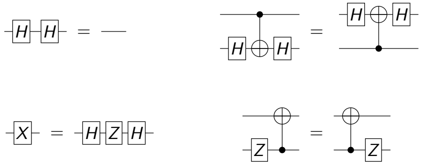

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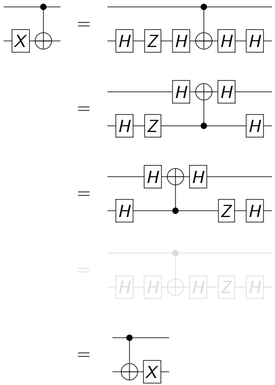

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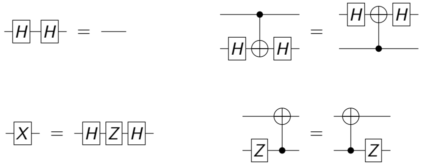

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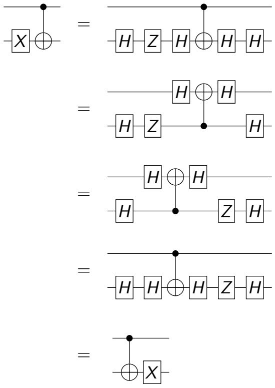

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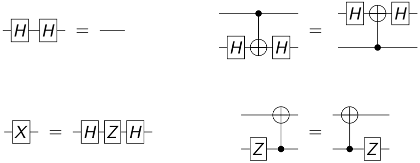

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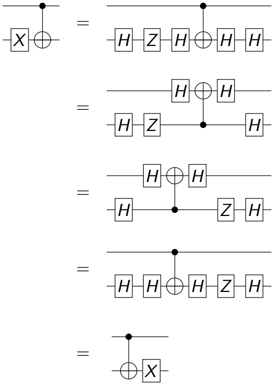

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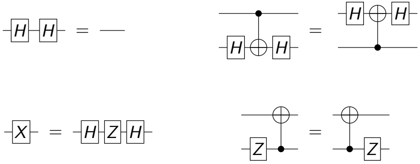

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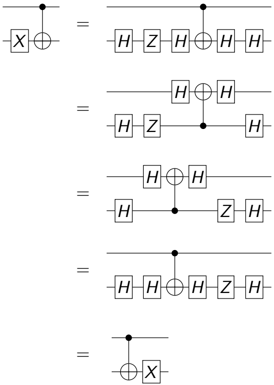

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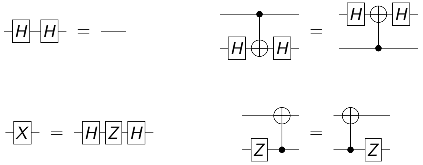

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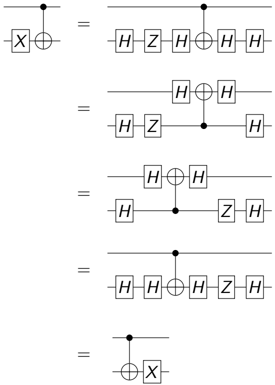

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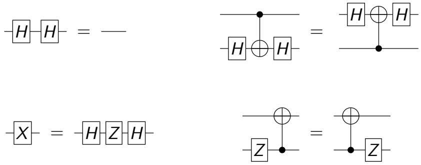

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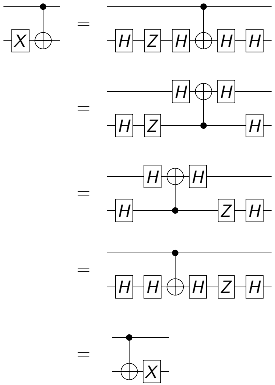

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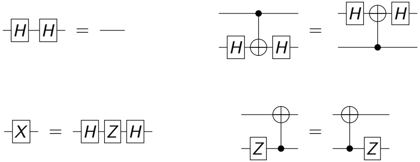

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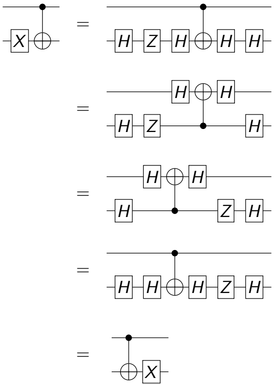

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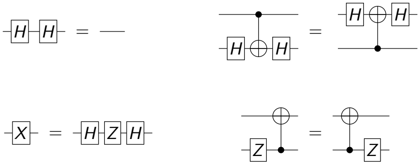

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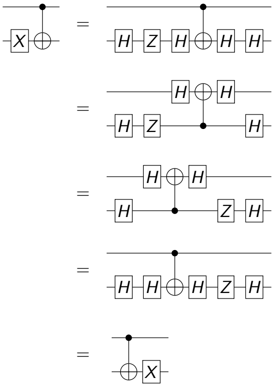

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Quantum Circuits as a Graphical Language

We consider the PROP of quantum circuits generated by:

global phase

phase gate ($\propto$ Z-rotation)

Hadamard

controlled not

Structure of PROP:

swap gate

sequential composition $C_2 \circ C_1$

parallel composition $C_1 \otimes C_2$

+ a few axioms, e.g.

"only connectivity matters"

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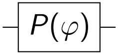

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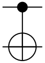

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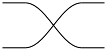

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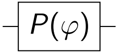

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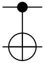

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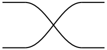

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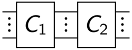

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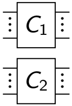

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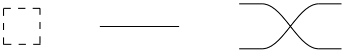

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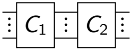

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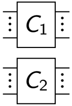

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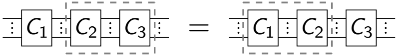

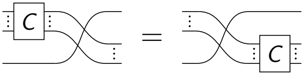

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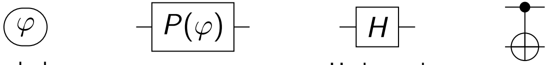

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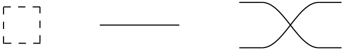

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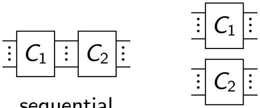

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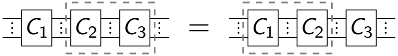

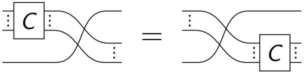

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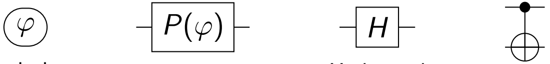

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$$-\boxed{Z}- \;:=\; -\boxed{P(\pi)}-$$

$$-\boxed{X}- \;:=\; -\boxed{H}-\boxed{Z}-\boxed{H}-$$

$$-\boxed{R_X(\theta)}- \;:=\; \left({}^{-\theta}/_{2}\right) -\boxed{H}-\boxed{P(\theta)}-\boxed{H}-$$

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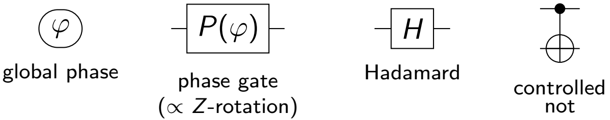

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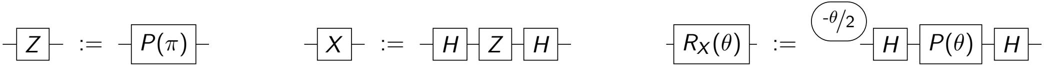

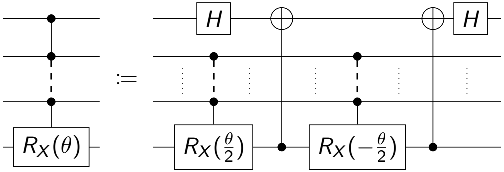

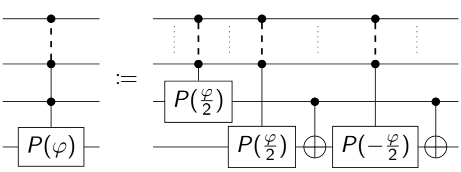

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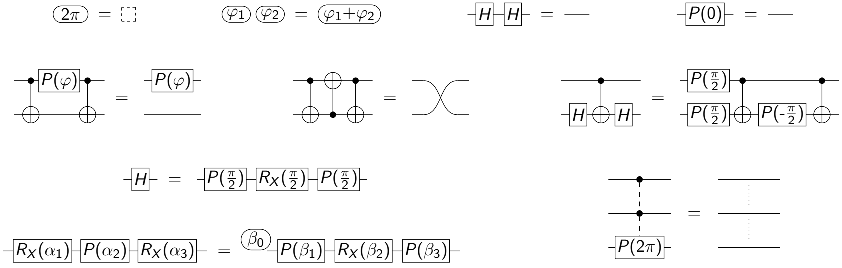
A Minimal Complete Equational Theory for Quantum Circuits



Theorem (Completeness [C., Delorme, Perdrix, 2023])

Any two equivalent circuits can be transformed into each other.

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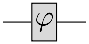
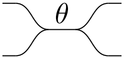


phase shifter



beam splitter

$e^{i\varphi}$

$$\begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix}$$

Structure of PROP:

sequential composition

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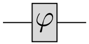

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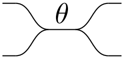

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A Graphical Language for Polarisation-Preserving Linear Optical Circuits

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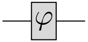

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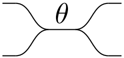

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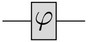

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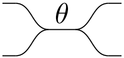

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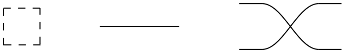
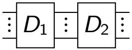
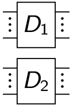




sequential composition

$D_2 \circ D_1$

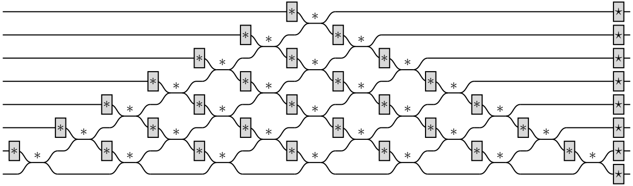


parallel composition

$D_1 \oplus D_2$

+ "only connectivity matters"

Universal form:



M. Reck, A. Zeilinger, H. J. Bernstein, P. Bertani, *Experimental realization of any discrete unitary operator*, 1994.

Proposition (Universality)

For any unitary $U: \mathbb{C}^n \to \mathbb{C}^n$, there exists a $\mathrm{LO}_{\mathrm{PP}}$-circuit $D: n \to n$ such that $[\![D]\!] = U$.

Unique if we impose conditions on the parameters $\longrightarrow$ **Normal form**

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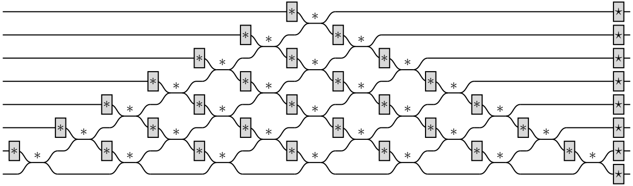

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A Graphical Language for Polarisation-Preserving Linear Optical Circuits

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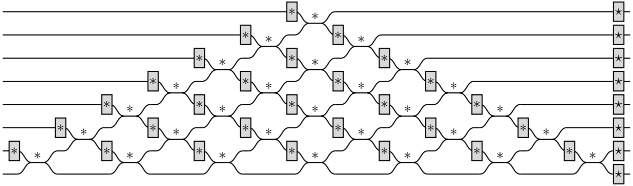

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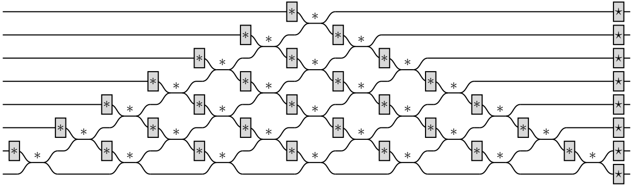

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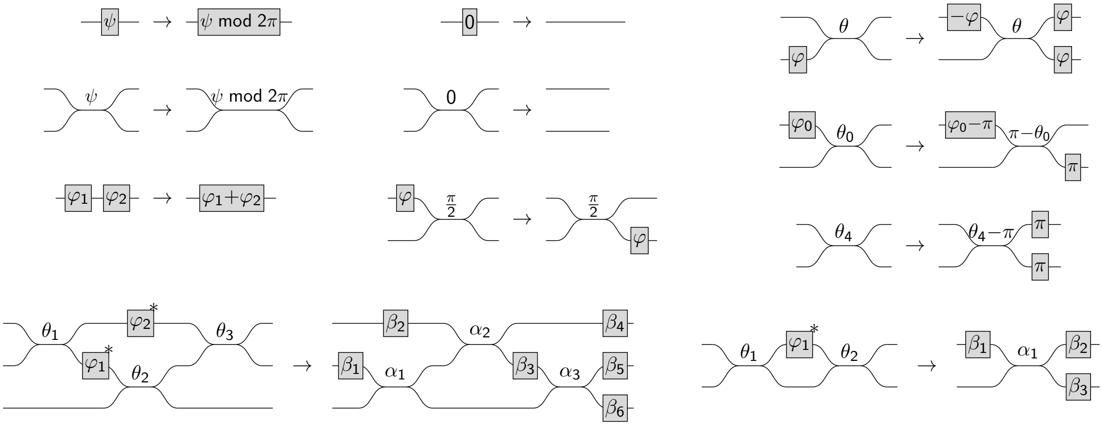
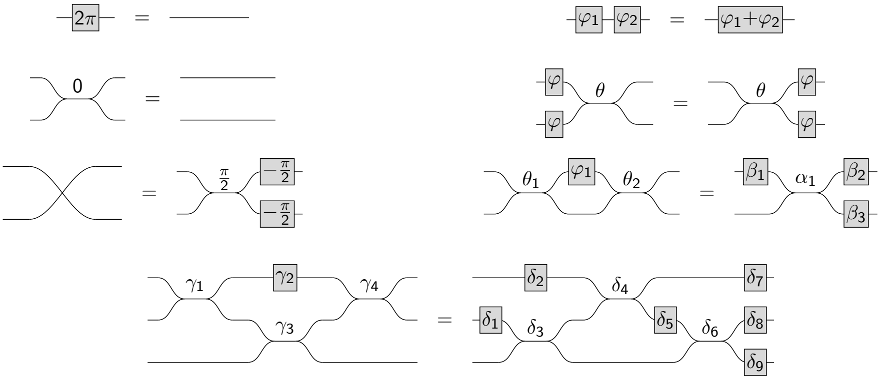
A Rewriting System for $\mathrm{LO}_{\mathrm{PP}}$-Circuits



where $\psi \in \mathbb{R} \setminus [0, 2\pi)$; $\varphi, \varphi_1, \varphi_2 \in (0, 2\pi)$; $\varphi_0, \theta_4 \in [\pi, 2\pi)$; $\theta, \theta_0, \theta_1, \theta_2, \theta_3 \in (0, \pi)$; and $\theta_0 \neq \frac{\pi}{2}$.

A Complete Equational Theory for $\mathrm{LO}_{\mathrm{PP}}$-Circuits



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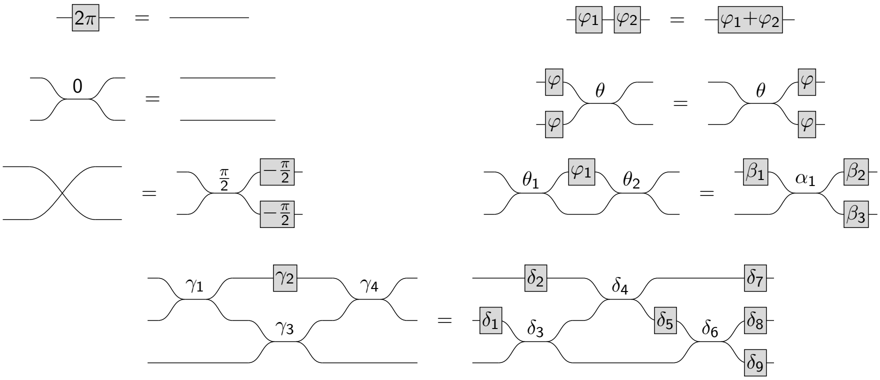

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Proof of Completeness for Quantum Circuits

Remark

The 2^n-mode $\mathrm{LO}_{\mathrm{PP}}$-circuits are universal for $2^n \times 2^n$ unitaries, like the n-qubit quantum circuits.

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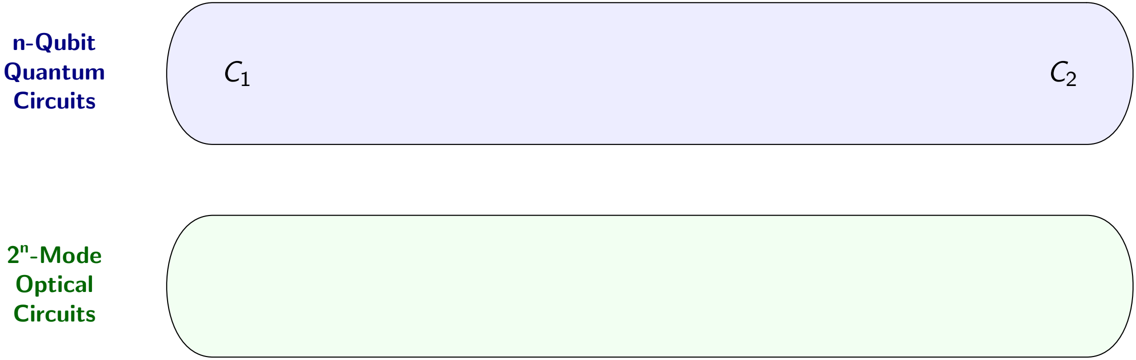

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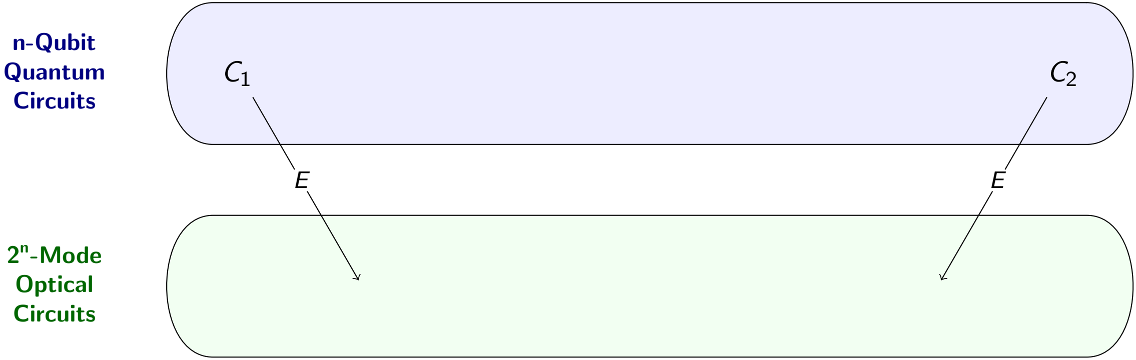

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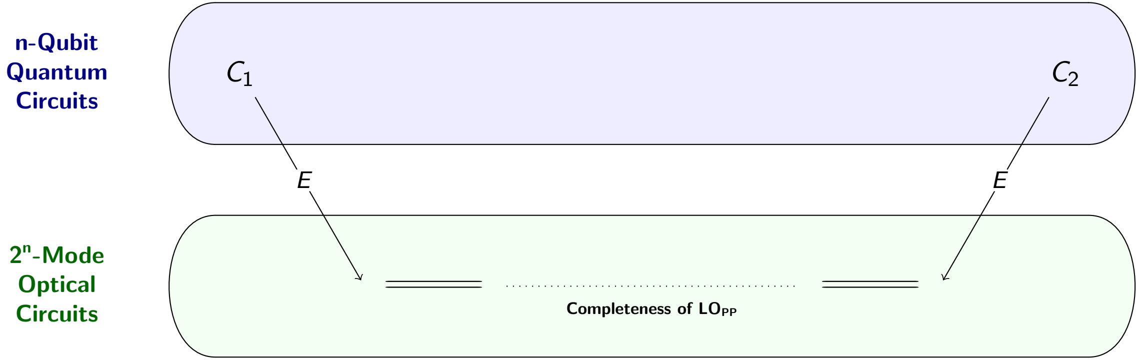

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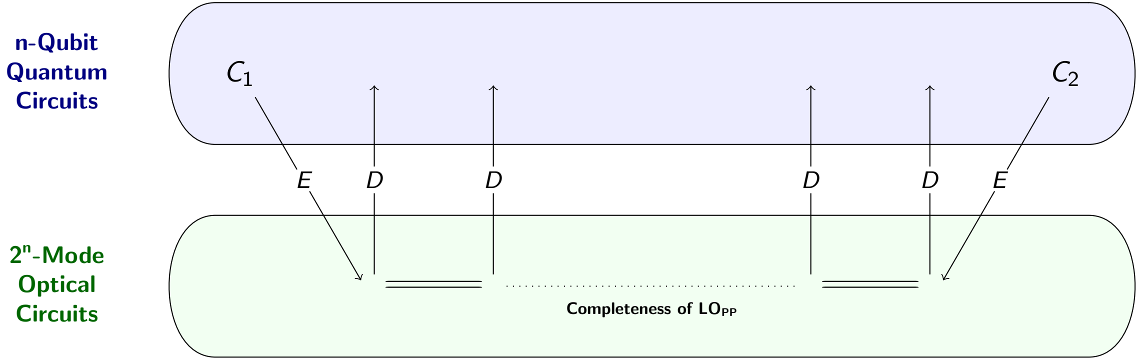


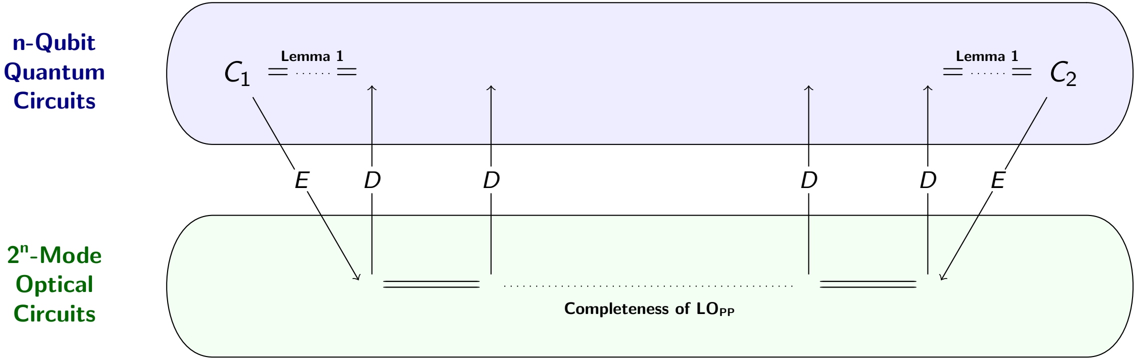
Proof of Completeness for Quantum Circuits

Lemma 1

For any quantum circuit C, $\mathrm{QC} \vdash D(E(C)) = C$.

Lemma 2

For every equation of the $\mathrm{LO_{PP}}$*-calculus, of the form* $D_1 = D_2$*, one has* $\mathrm{QC} \vdash D(D_1) = D(D_2)$.



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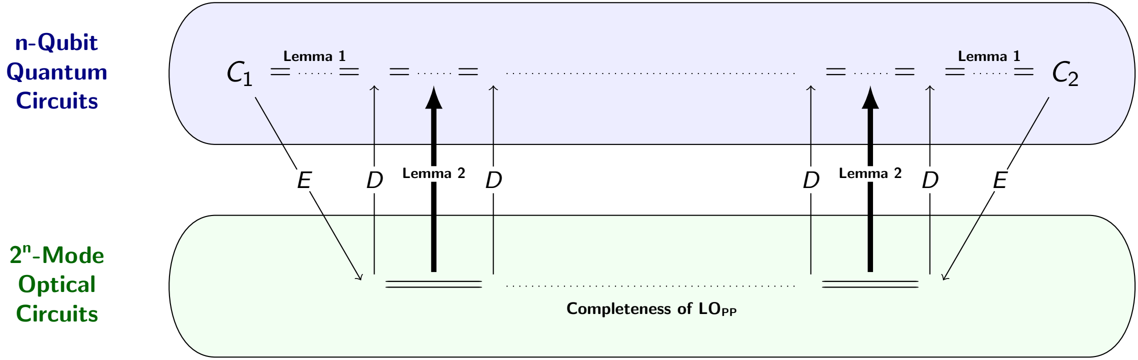
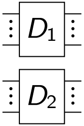


Key Aspects of the Proof

- Note that optical circuits and quantum circuits have different structures:

Optical circuits:



$D_1 \oplus D_2$

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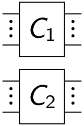

$C_1 \otimes C_2$

$\Rightarrow$ To define E and D, we "sequentialise" the circuits:

$\Rightarrow$ Some deformation rules need to be treated as proper equations.

- Decoding produces multi-controlled gates, e.g.:

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two-level matrix

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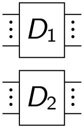

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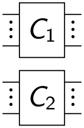

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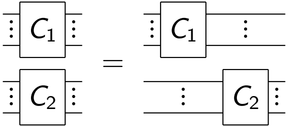

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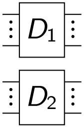

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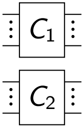

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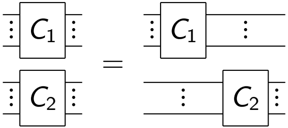

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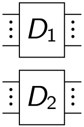

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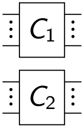

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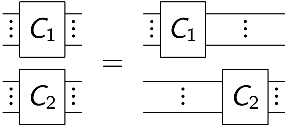

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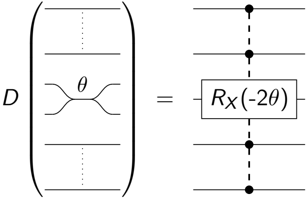

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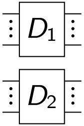

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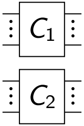

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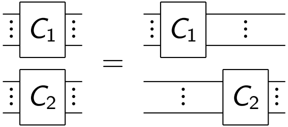

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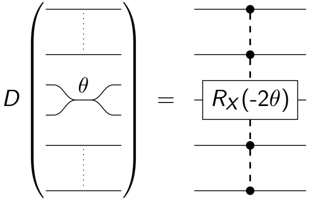

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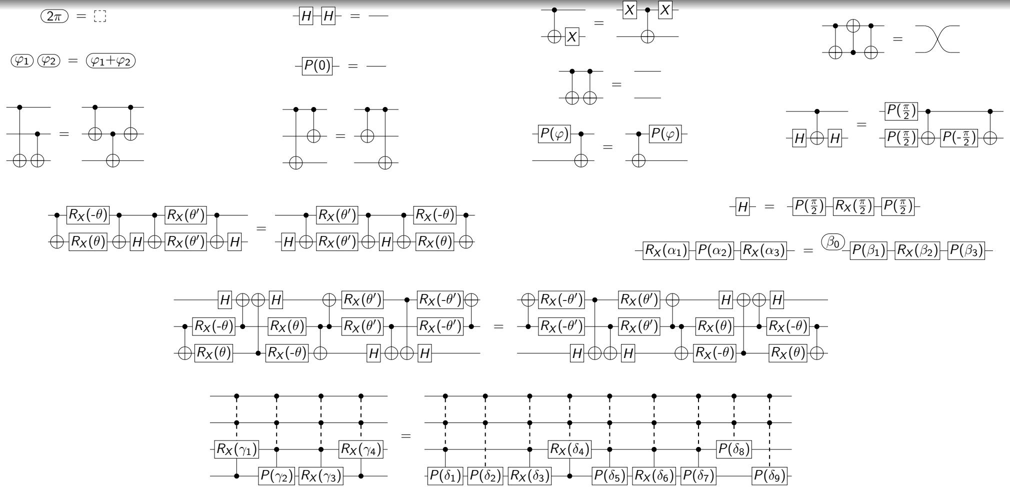
A Complete Equational Theory for Quantum Circuits



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Any two equivalent circuits can be transformed into each other.

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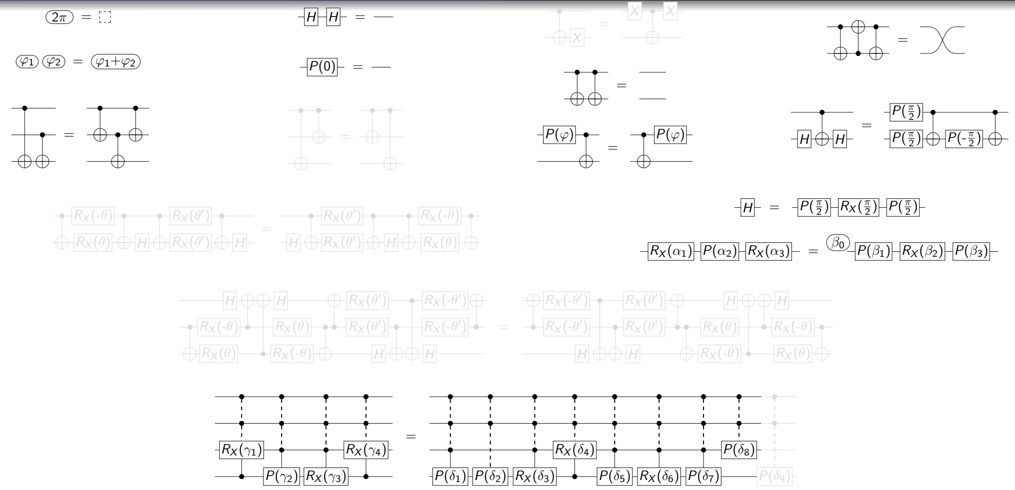

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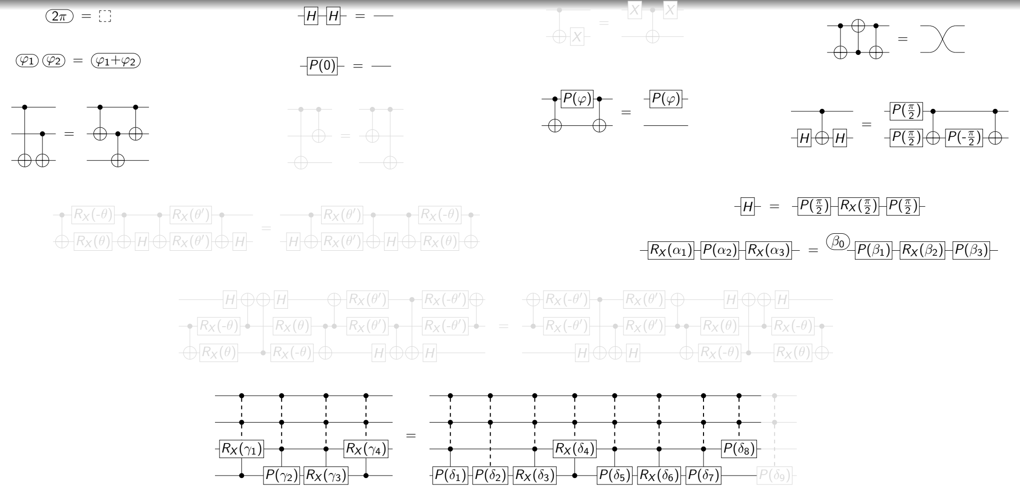

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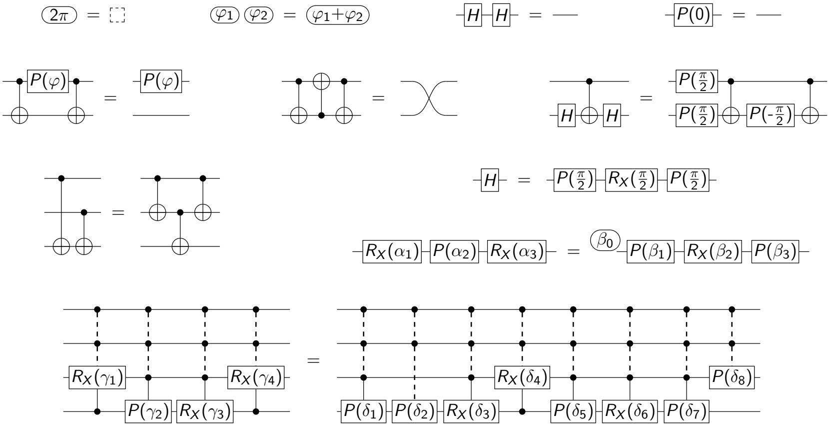

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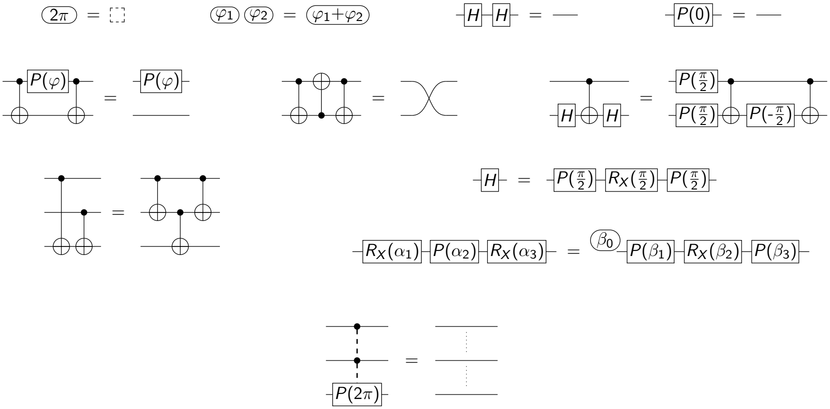

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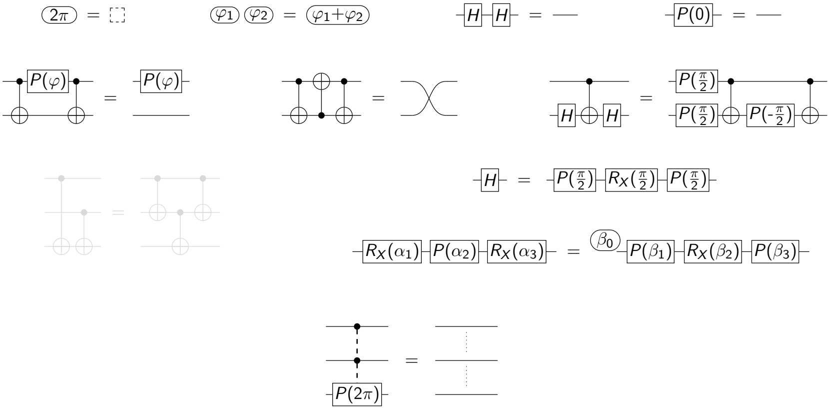

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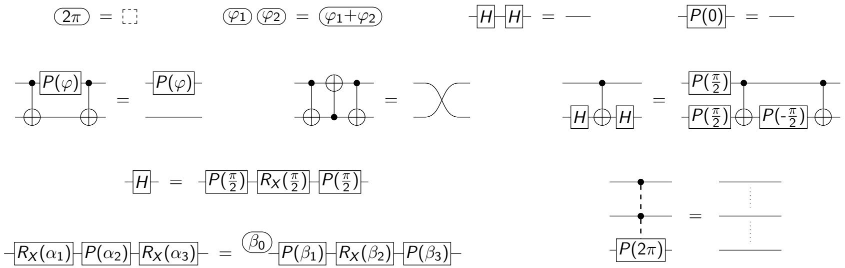

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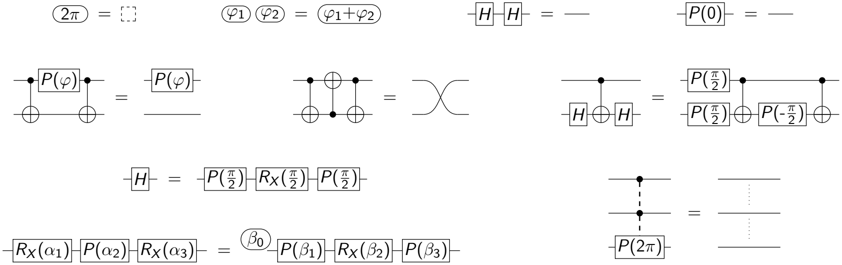

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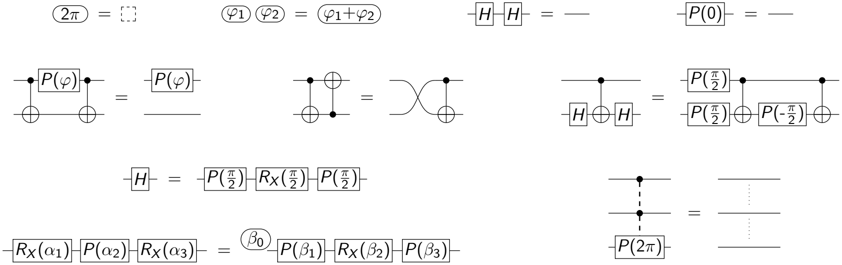

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Minimality: Necessity of Equations on Any Number of Qubits

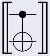
Definition (Alternative Interpretation)

For any $k \in \mathbb{N}$, let $[\![C]\!]^\sharp_k \in [0, 2\pi)$ be inductively defined $(\mathrm{mod}\, 2\pi)$ as

$$[\![C_2 \circ C_1]\!]^\sharp_k = [\![C_2]\!]^\sharp_k + [\![C_1]\!]^\sharp_k \qquad [\![\textcircled{\varphi}]\!]^\sharp_k = 2^k\varphi$$

[[]]$^\sharp_k$ = [[—]]$^\sharp_k$ = 0

$$[\![C_1 \otimes C_2]\!]^\sharp_k = [\![C_1]\!]^\sharp_k + [\![C_2]\!]^\sharp_k \qquad [\![-P(\varphi)-]\!]^\sharp_k = 2^{k-1}\varphi \qquad [\![-H-]\!]^\sharp_k = 2^{k-1}\pi$$

[[]]$^\sharp_k = 2^{k-2}\pi$ [[]]$^\sharp_k = 2^{k-2}\pi$

Proposition

$\forall k \geq n, \quad e^{i[\![C]\!]^\sharp_k} = \det([\![C]\!])^{2^{k-n}}$

Corollary

For any $C_1, C_2 : n \to n$ and any $k \geq n$, if $[\![C_1]\!] = [\![C_2]\!]$ then $[\![C_1]\!]^\sharp_k = [\![C_2]\!]^\sharp_k$.

Corollary

Any equational theory on at most k qubits is sound with respect to $[\![\cdot]\!]^\sharp_k$.

[[$: n \to n$]]$^\sharp_k = 2^{1+k-n}\pi$

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$$= 2^{1+k-n}\pi$$

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For any $k \in \mathbb{N}$, let $\llbracket C \rrbracket_k^\sharp \in [0, 2\pi)$ be inductively defined $(\bmod 2\pi)$ as

$$\llbracket C_2 \circ C_1 \rrbracket_k^\sharp = \llbracket C_2 \rrbracket_k^\sharp + \llbracket C_1 \rrbracket_k^\sharp \qquad \llbracket \textcircled{\varphi} \rrbracket_k^\sharp = 2^k \varphi \qquad \llbracket \; \rrbracket_k^\sharp = \llbracket - \rrbracket_k^\sharp = 0$$

$$\llbracket C_1 \otimes C_2 \rrbracket_k^\sharp = \llbracket C_1 \rrbracket_k^\sharp + \llbracket C_2 \rrbracket_k^\sharp \qquad \llbracket \boxed{P(\varphi)} \rrbracket_k^\sharp = 2^{k-1}\varphi \qquad \llbracket \boxed{H} \rrbracket_k^\sharp = 2^{k-1}\pi$$

$$\llbracket \mathrm{CNOT} \rrbracket_k^\sharp = 2^{k-2}\pi \qquad \llbracket \mathrm{SWAP} \rrbracket_k^\sharp = 2^{k-2}\pi$$

Proposition

$$\forall k \geq n, \quad e^{i\llbracket C \rrbracket_k^\sharp} = \det(\llbracket C \rrbracket)^{2^{k-n}}$$

Corollary

For any $C_1, C_2 : n \to n$ and any $k \geq n$, if $\llbracket C_1 \rrbracket = \llbracket C_2 \rrbracket$ then $\llbracket C_1 \rrbracket_k^\sharp = \llbracket C_2 \rrbracket_k^\sharp$.

Corollary

Any equational theory on at most k qubits is sound with respect to $\llbracket \cdot \rrbracket_k^\sharp$.

$$\left\llbracket\right.$$ $$: n \to n \left.\right\rrbracket_k^\sharp = 2^{1+k-n}\pi$$

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For any $k \in \mathbb{N}$, let $[\![C]\!]^\sharp_k \in [0, 2\pi)$ be inductively defined $(\mathrm{mod} 2\pi)$ as

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Minimality: Necessity of Equations on Any Number of Qubits

Definition (Alternative Interpretation)

For any $k \in \mathbb{N}$, let $[\![C]\!]^\sharp_k \in [0, 2\pi)$ be inductively defined $(\bmod 2\pi)$ as

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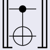
$[\![$  $]\!]^\sharp_k = 2^{k-2}\pi$ $\qquad$ $[\![$  $]\!]^\sharp_k = 2^{k-2}\pi$

Proposition

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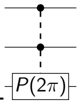
$[\![$  $: n \to n]\!]^\sharp_k = 2^{1+k-n}\pi$

Minimality: Necessity of Equations on Any Number of Qubits

Theorem

Any complete equational theory on vanilla quantum circuits contains an equation on n qubits for every n.

Remark

This result remains true if

- we consider circuits up to global phases
- we change the set of generators.

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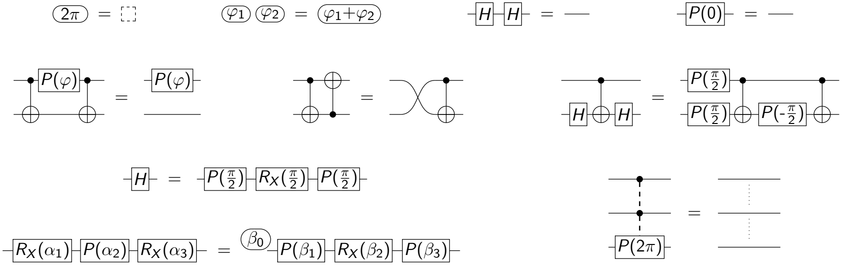
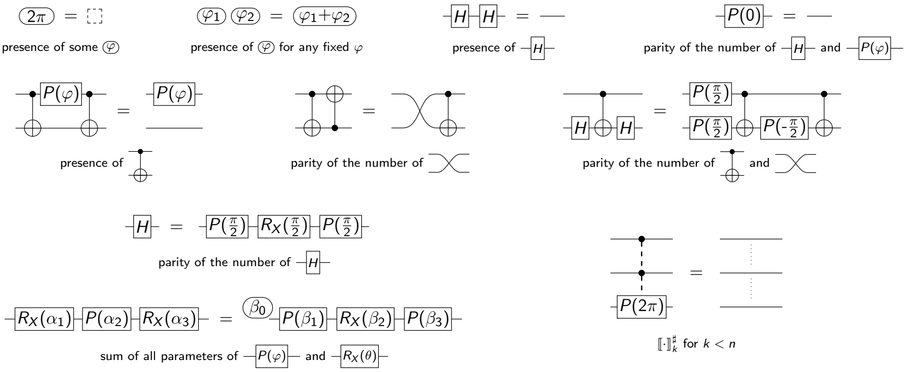


Theorem (Minimality)

None of these equations can be derived from the others.

Minimality



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None of these equations can be derived from the others.

Circuits With Ancillae

preparation
(i.e. initialisation)

release

Circuits With Ancillae

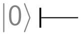
$|0\rangle \vdash$

preparation
(i.e. initialisation)

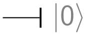
$\dashv |0\rangle$

release

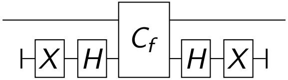
Circuits With Ancillae



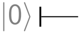
preparation
(i.e. initialisation)



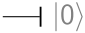
release



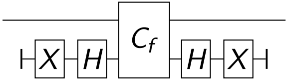
phase oracle



preparation
(i.e. initialisation)



release



phase oracle

universal for isometries

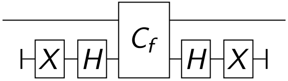
Circuits With Ancillae

$|0\rangle \vdash\!\!-$

preparation
(i.e. initialisation)

$-\!\!\dashv |0\rangle$

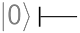
release



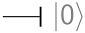
phase oracle

universal for isometries

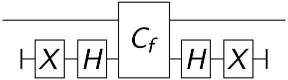
(restriction to isometries)



preparation
(i.e. initialisation)

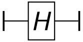


release



phase oracle

universal for isometries

(restriction to isometries: 

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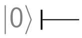

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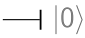
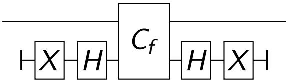
(i.e. initialisation)

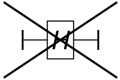


release



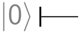
phase oracle

universal for isometries

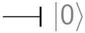
(restriction to isometries: 

)

Circuits With Ancillae or Trace Out



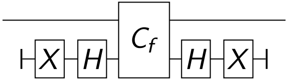
preparation
(i.e. initialisation)



release

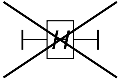
or

partial trace
i.e. trace out
i.e. discard



phase oracle

universal for isometries

(restriction to isometries: 

)

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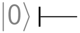
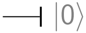


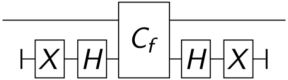
preparation
(i.e. initialisation)



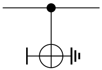
release

or

partial trace
i.e. trace out
i.e. discard

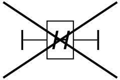


phase oracle



measurement

universal for isometries

(restriction to isometries: 

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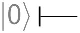
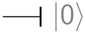


preparation
(i.e. initialisation)



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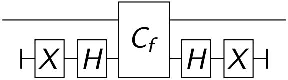
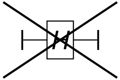
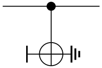


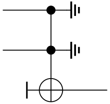
phase oracle

universal for isometries

(restriction to isometries: )

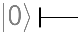


measurement

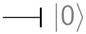


classical
AND gate

Circuits With Ancillae or Trace Out



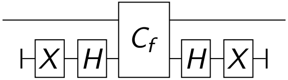
preparation
(i.e. initialisation)



release

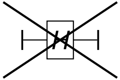
or

partial trace
i.e. trace out
i.e. discard



phase oracle

universal for isometries

(restriction to isometries: 

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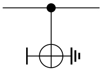
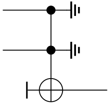


measurement

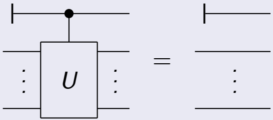


classical
AND gate

universal for CPTP maps

Lemma

Given a complete equational theory for vanilla quantum circuits, adding all equations of the following form makes it complete for quantum circuits with $\vdash$:



Proof using cosine-sine decomposition or following S. Staton, *Algebraic Effects, Linearity, and Quantum Programming Languages*, 2015.

Corollary

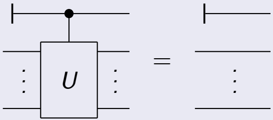
Given a complete equational theory for vanilla quantum circuits, adding the following two equations makes it complete for quantum circuits with $\vdash$:

Shortcuts:

Complete Equational Theories for Circuits With Auxiliary Qubits

Lemma

Given a complete equational theory for vanilla quantum circuits, adding all equations of the following form makes it complete for quantum circuits with ⊢:



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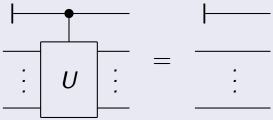
Complete Equational Theories for Circuits With Auxiliary Qubits

Lemma

Given a complete equational theory for vanilla quantum circuits, adding all equations of the following form makes it complete for quantum circuits with $\vdash$*:*

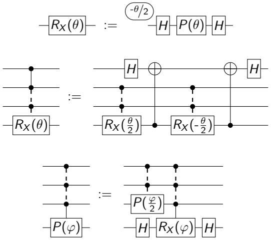


Proof using cosine-sine decomposition or following S. Staton, *Algebraic Effects, Linearity, and Quantum Programming Languages*, 2015.

Corollary

Given a complete equational theory for vanilla quantum circuits, adding the following two equations makes it complete for quantum circuits with $\vdash$*:*

Shortcuts:



Complete Equational Theories for Circuits With Auxiliary Qubits

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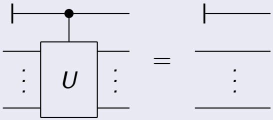

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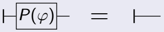
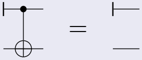
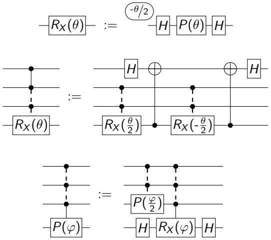
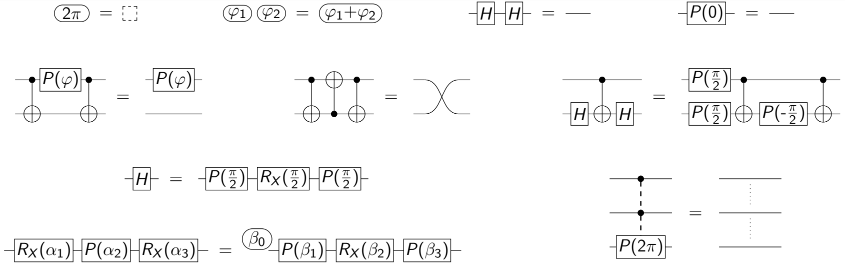




Shortcuts:



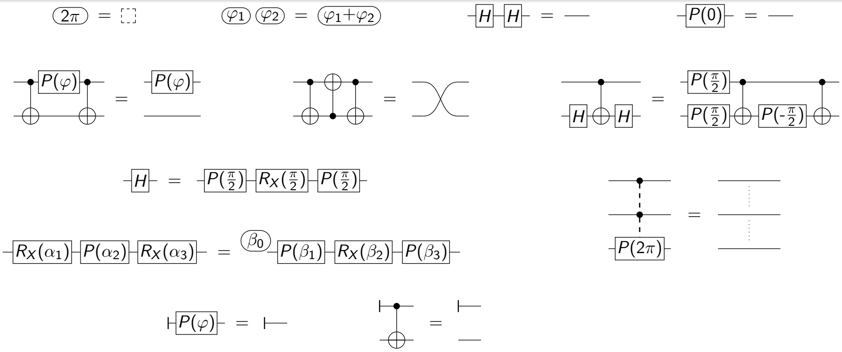
Circuits With Initialisations (⊢)



Theorem (Completeness)

Any two equivalent circuits can be transformed into each other.

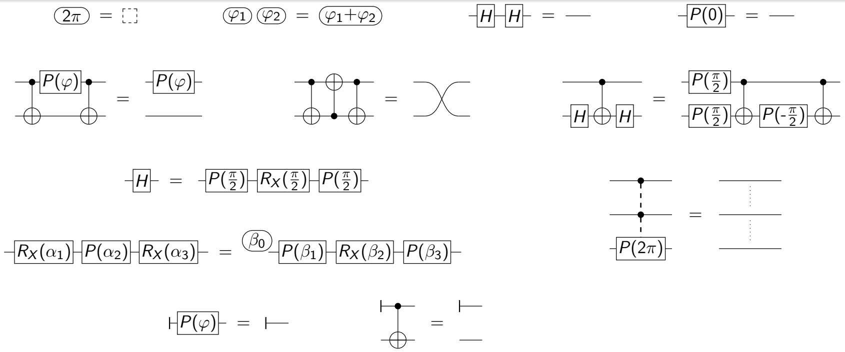
Circuits With Initialisations (⊢)



Theorem (Completeness)

Any two equivalent circuits can be transformed into each other.

Circuits With Ancillae (⊢ and ⊣)



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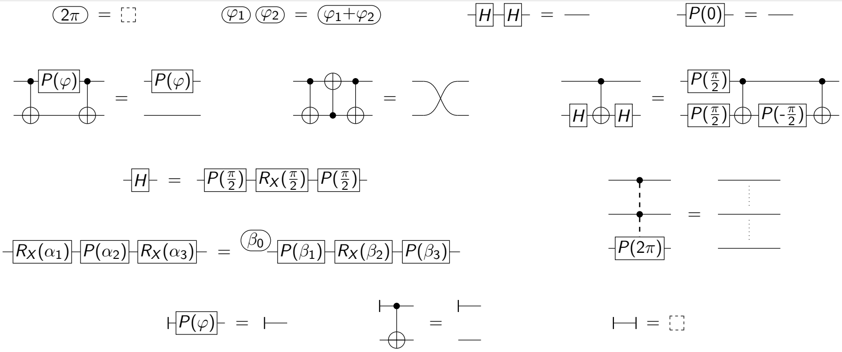

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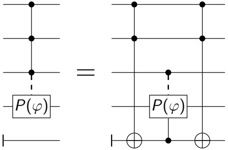

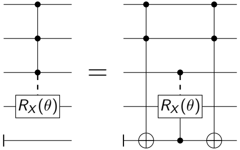

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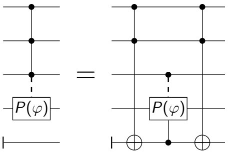

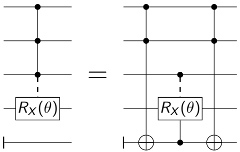

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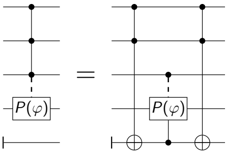

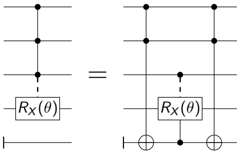

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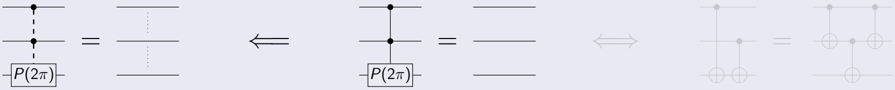

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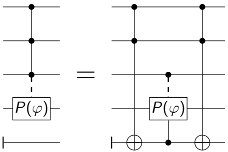

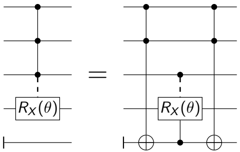

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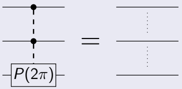

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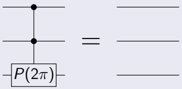

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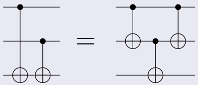

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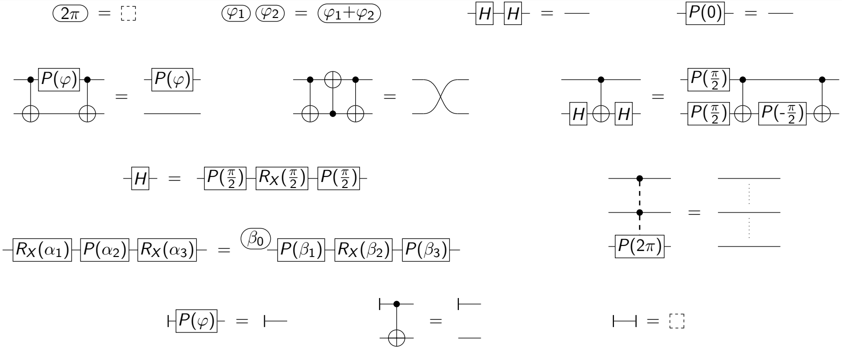

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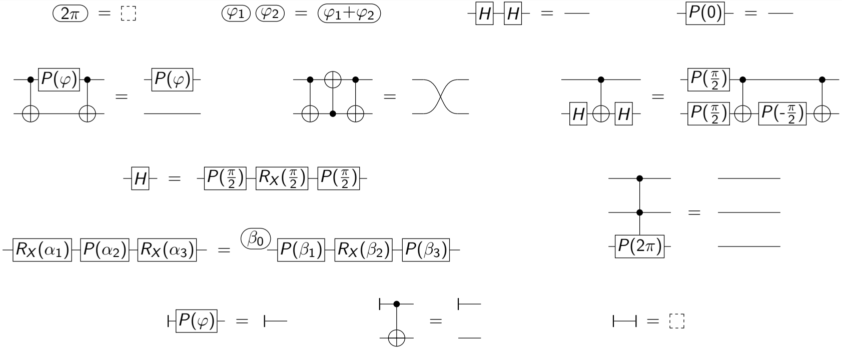

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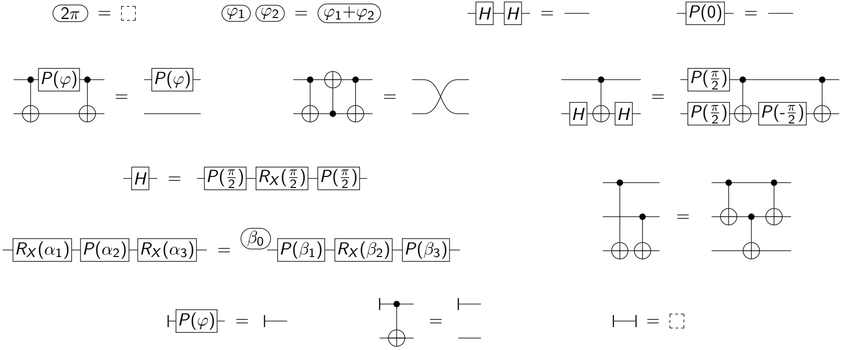

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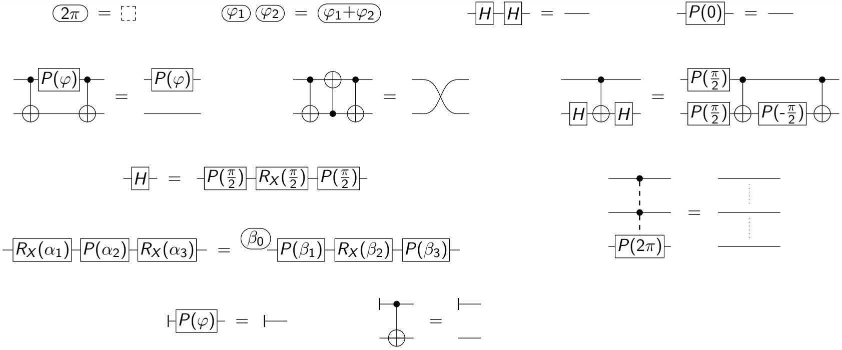

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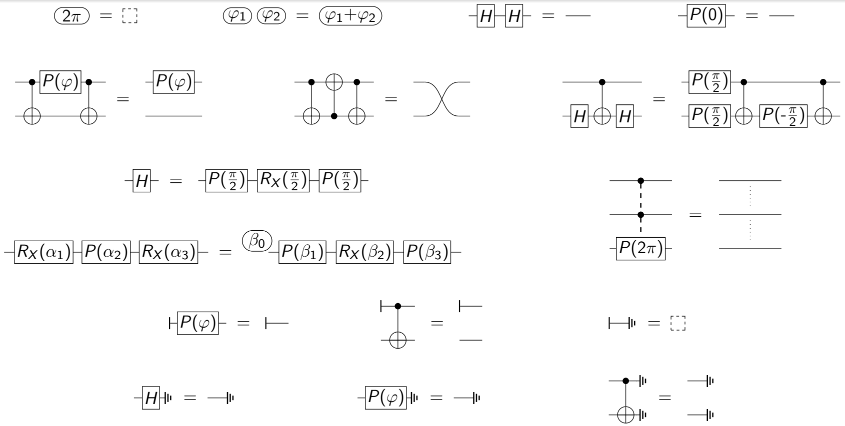

Carette, Jeandel, Perdrix, Vilmart, *Completeness of Graphical Languages for Mixed States Quantum Mechanics*, 2019, or Huot and Staton, *Quantum Channels as a Categorical Completion*, 2019.

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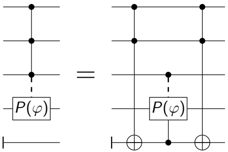

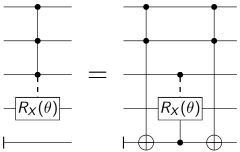

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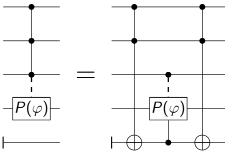

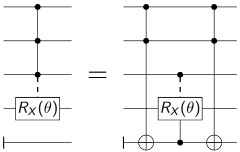

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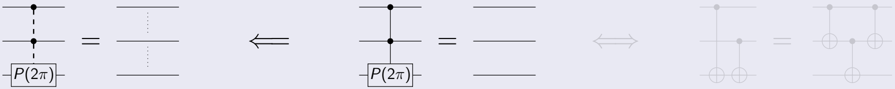

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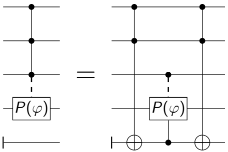

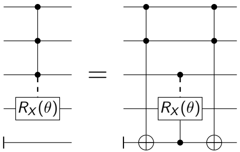

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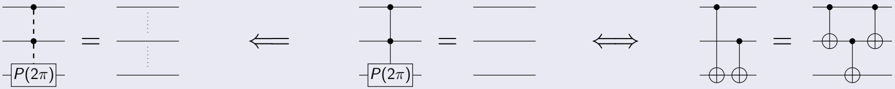

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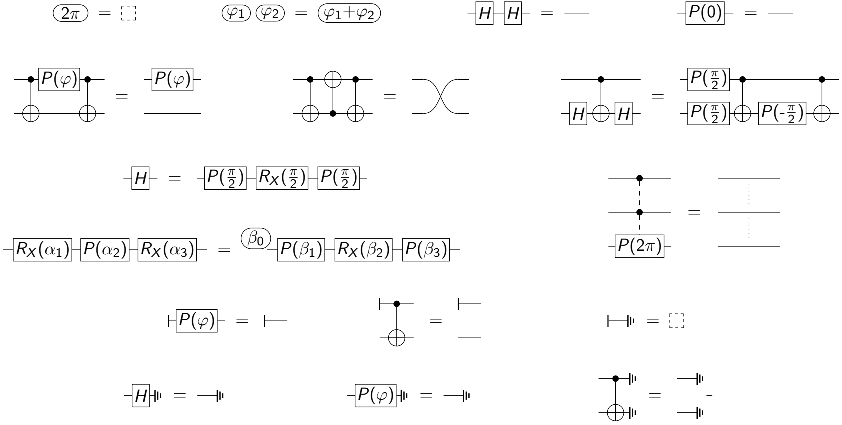

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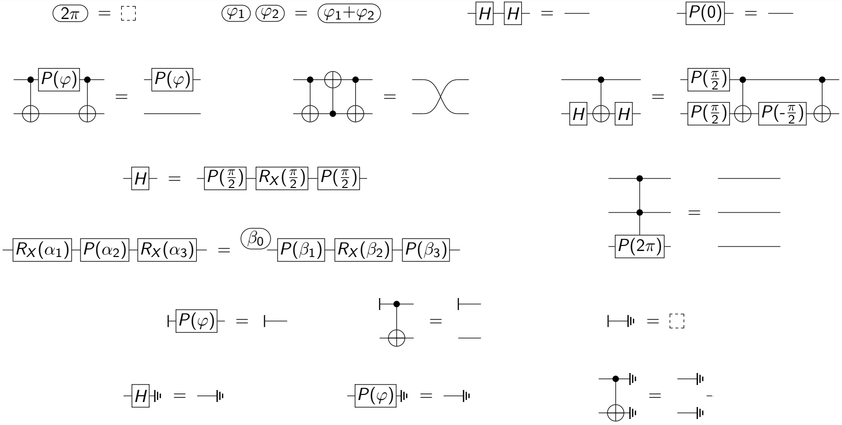

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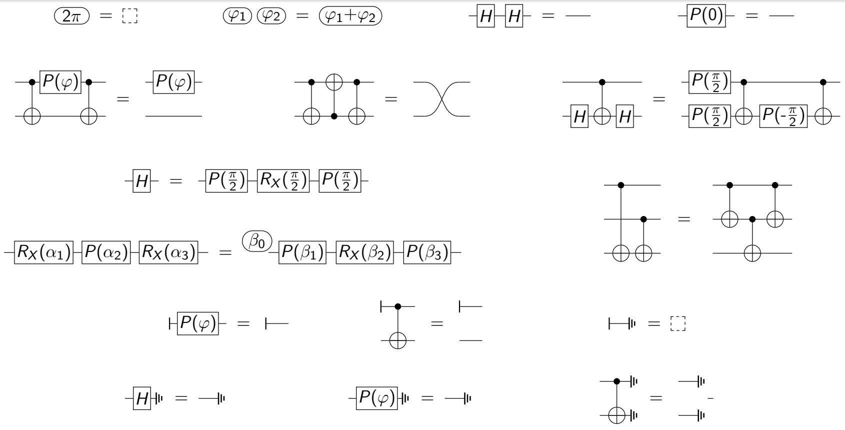

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Some Ongoing Follow-Up Work

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  - Subfields of $\mathbb{C}$ or $\mathbb{R}$
    - **Theorem:** the sets of angles closed under $-R_X(\alpha_1)-P(\alpha_2)-R_X(\alpha_3)- = \beta_0\, -P(\beta_1)-R_X(\beta_2)-P(\beta_3)-$ correspond to fields of constructible numbers. $\rightarrow$ Any usefulness for practical applications ?
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Conclusion and Future Works

Summary of the results:

- A minimal complete equational theory for vanilla quantum circuits.
- First minimality result for a quantum graphical language (except PBS-calculus).
  - Shortly after, minimality for qudit ZW-calculus (R. Vilmart and M. de Visme, January 2024).
- One needs equations on an unbounded number of qubits.
- Simple equational theories for circuits with ancillae and discard, with equations on at most 3 qubits.

Some perspectives:

- Ongoing work: Clifford+T, other fields/constructible angles, qudits.
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