Introduction to Automata Networks

Kévin Perrot

CANA seminar

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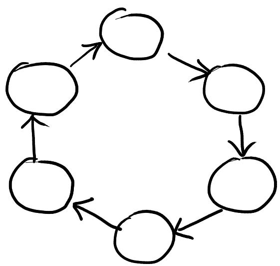
Definitions

$n \in \mathbb{N}$, $[n]=\{1,\dots,n\}$, $X = \prod_{i\in[n]} A_i$, $f : X \to X$.

- LOCAL FUNCTIONS $f_i : X \to A_i$.
- INTERACTION GRAPH $G_f = (V, A)$ with $V = [n]$
  AND $(i,j) \in A \Leftrightarrow j$ EFFECTIVELY DEPENDS ON i.
  $\exists x \in X : f_j(x) \neq f_j(x + e_i)$
- DYNAMICS (PARALLEL) $\mathcal{G}_f = (X, \{(x, f(x)) \mid x \in X\})$.

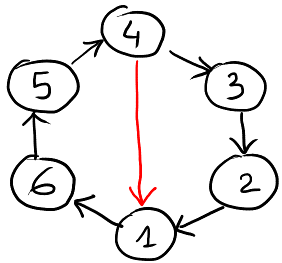
FIXED POINT $x \in X$ s.t. $f(x) = x$.

$G_f =$



$n=6$, $X = \{0,1\}^n$, $f_i(x) = x_{(i \bmod n)+1}$

#FP(f) = ?



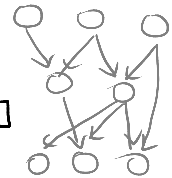
- $f_1(x) = x_2 \vee \neg x_4$

  #FP(f) = ?
- $f_1(x) = x_2 \wedge \neg x_4$

  #FP(f) = ?

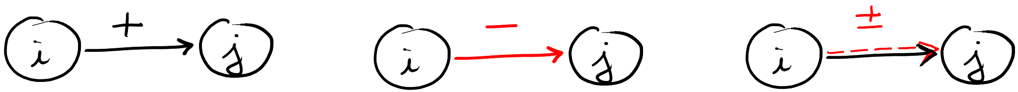
LAWS

LAW OF ROBERT (1976): G_f ACYCLIC $\Longrightarrow$ f^n CONSTANT.



PROOF: TOPOLOGICAL SORT, BY INDUCTION AT TIME t THE AUTOMATA AT DISTANCE $t-1$ FROM A SOURCE ARE CONSTANT. $\square$

SIGNED G_f



CYCLE + (EVEN NUMBER OF − ARCS), CYCLE − (ODD NUMBER OF − ARCS)

♥ FIXED POINTS

LAW OF THOMAS (1981, COMET-RICHARD 2007, NOUAL-SENÉ 2012):

$\#FP(f) \geq 2 \Longrightarrow G_f$ HAS A CYCLE +.

(PROOF NEXT SLIDE FOR f UNATE, IN PARALLEL.)

BOUNDING $\mathrm{MaxFP}(G) = \max\{\#FP(f) \mid f \text{ s.t. } G_f = G\}$.

THM (ARACENA 2008): FOR UNATE f_i, $\mathrm{MaxFP}(G) \leq 2^{\tau^+(G)}$

UNATE = MONOTONOUS LOCAL FUNCTIONS $(x \leq_{\pm} y \Rightarrow f_i(x) \leq f_i(y))$.

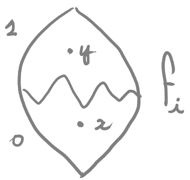
$\tau^+(G)$ = POSITIVE TRANSVERSAL NUMBER (MINIMUM FVS+).

PROOF: LET f UNATE S.T. $G_f = G$.

TWO FIXED POINTS $x \neq y$ MUST DIFFER ON A WHOLE CYCLE + :

UNATE PROPERTY:

WLOG: $x_i = 0$, $y_i = 1$.



IF $\forall j$, $j \xrightarrow{+} i$: $x_j \geq y_j$

AND $\forall j$, $j \xrightarrow{-} i$: $x_j \leq y_j$

THEN $f_i(x) \geq f_i(y)$.

SO i MUST HAVE AN IN-NEIGHBOR j COHERENT WITH ITS SIGN: $j\,(0/1) \xrightarrow{+} i\,(0/1)$ OR $j\,(1/0) \xrightarrow{-} i\,(0/1)$.

BACKWARDS, WE MUST CLOSE A CYCLE ON 0/1 AGAIN (EVEN PARITY OF $-$).

THUS g: FIXED POINT $x \longmapsto x_F$ WITH F A FVS+, IS INJECTIVE.

$$\#FP(f) \leq 2^{|F|}.$$

$\square$

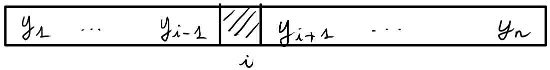
BOUNDED DEGREE $\Delta(G_f) \leq d$ (TRULY LOCAL).

q-UNIFORM ($\forall i \in [n]$: $A_i = \{0, \dots, q-1\}$).

THM (2023): $f \neq$ IDENTITY S.T. $\Delta(G_f) \leq d \implies \#FP(f) \leq q^n - q^{n-d}$.

PROOF: $\exists x : f_i(x) \neq x_i$.

- IF $(i,i) \notin G_f$ THEN

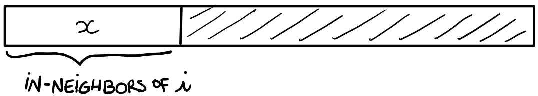


$\in X$

$\forall y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n : \exists!\, y_i : f_i(y) = y_i$

SO $\#FP(f) \leq q^{n-1} \leq q^n - q^{n-d}$.

- IF $(i,i) \in G_f$ THEN



$\in X$

$\forall y$ S.T. $y_j = x_j$ FOR ALL $(j,i) \in G_f$

WE HAVE $f_i(y) = f_i(x) \neq x_i = y_i$ I.E. $f(y) \neq y$.

SO $\#FP(f) \leq q^n - q^{n-d}$. $\square$

FIXED POINTS INVARIANCE

BLOC-SEQUENTIAL UPDATE MODES : ORDERED PARTITIONS OF $[n]$.

E.G. $(\{1,3\},\{4\},\{2,5,6\})$.

$|BS(n)|$ = ORDERED BELL NUMBERS $= \sum_{i=1}^{n} \binom{n}{i} \cdot |BS(n-i)|$ WITH $|BS(0)| = |BS(1)| = 1$.

THM (ROBERT 1986): FOR ANY f AND $\mu \in BS(n)$, $f(x) = x \iff f_\mu(x) = x$.

PROOF: BY INDUCTION ON THE SUBSTEPS OF μ, $f_i(x) = x_i$ FOR ALL $i \in [n]$. $\square$

LEAH'S PhD IS ON BLOC-PARALLEL UPDATE MODES

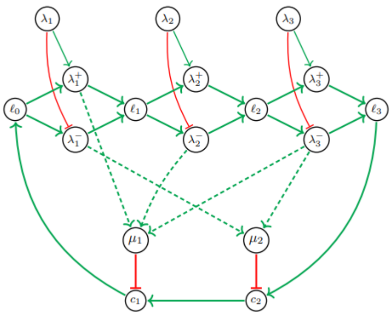
Complexity on input G_f (Boolean, parallel).

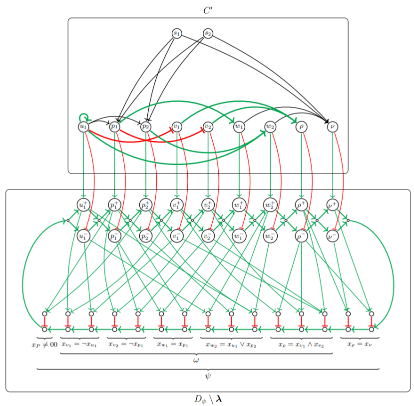
Prop: G is a signed interaction graph iff
$|\mathcal{N}^{\pm}_{in}(i)| \neq 1$ or $|\mathcal{N}_{in}(i)| \geq 3$ for all $i \in [n]$.

On n automaton, there are 2^{n2^n} BANs and 4^{n^2} signed graphs.
$\leftarrow \{\emptyset, +, -, \pm\}$

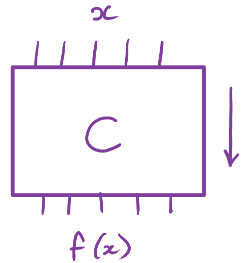
Thm (2019, 2022): Given G, deciding whether:

- $\text{MaxFP}(G) \geq k$ (fixed) is: in P for $k=1$ ($\Leftrightarrow \exists$ cycle $+$).
  NP-complete for any $k \geq 2$.
- $\text{MaxFP}(G) < k$ (fixed) is NEXPTIME-complete for any $k \geq 1$.





COMPLEXITY "À LA RICE"

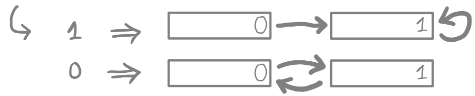


THM (ALON 1985): GIVEN f, DECIDING WHETHER $\exists x : x \to x$ IS NP-COMPLETE.

f IS ENCODED AS BOOLEAN FORMULAS OR MORE GENERALLY AS A CIRCUIT.

PROOF: GIVEN φ ON n VARIABLES, CONSTRUCT f ON $n+1$ BOOLEAN AUTOMATA WITH:

$f_i(x) = x_i$ FOR $i \in [n]$ AND $f_{n+1}(x) = \varphi(x_{[n]}) \vee \neg x_{n+1}$.



$\square$

THM (2021): ANY NON-TRIVIAL QUESTION EXPRESSIBLE IN F.O. IS NP-HARD OR coNP-HARD.

F.O.: LIMITED NUMBER OF QUANTIFIED CONFIGURATIONS, BUT ALTERNATIONS OF $\exists$ AND $\forall$.
(BIJECTIVITY, CONSTANT, ANY INDUCED SUBGRAPH ...)

ALIÉNOR'S PhD

THM (2025+): (ALMOST) ANY NON-TRIVIAL QUESTION EXPRESSIBLE IN M.S.O. IS NP-HARD OR coNP-HARD. (CONNECTIVITY, k-COLORABILITY ...)

ALSO FOR q-UNIFORM / NON-DETERMINISTIC. (CLIQUE, EXCLUDED MINOR ...)