Quantum measurement processes consistent with the second law of thermodynamics

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Self-introduction

- 1997-2016: I grew up in Nagoya, Japan
- April 2016-March 2020: Kyushu University, Japan (BSc, Earth science)
- April 2020- March 2025: Nagoya University, Japan (MPhil, PhD)
- April 2025: Tsukuba University, Japan
- May 2025- : Aix-Marseille University

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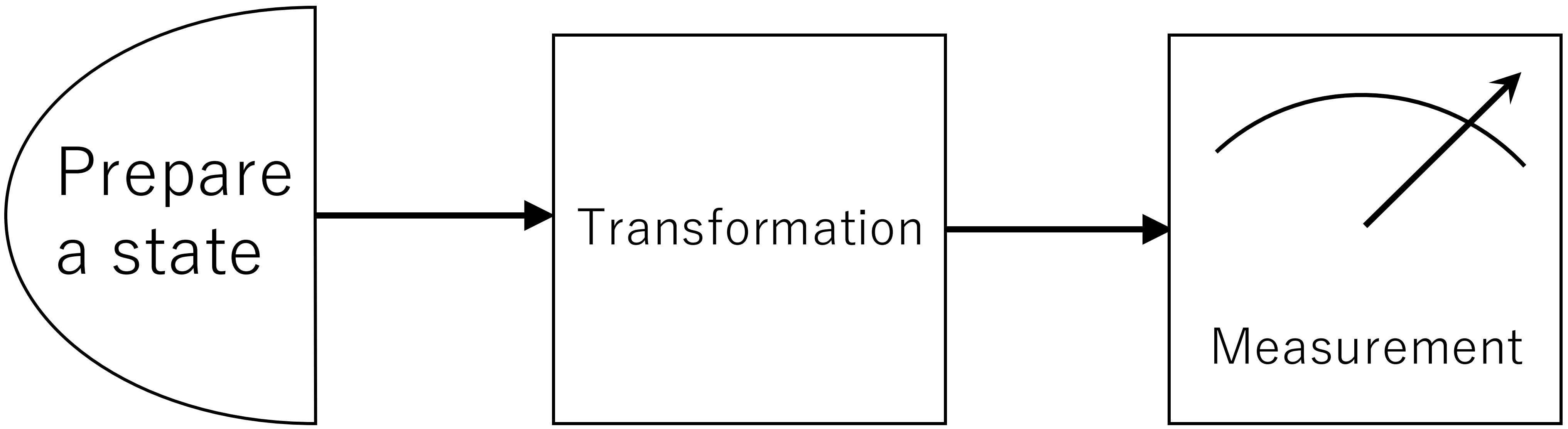
(irasutoya.com)

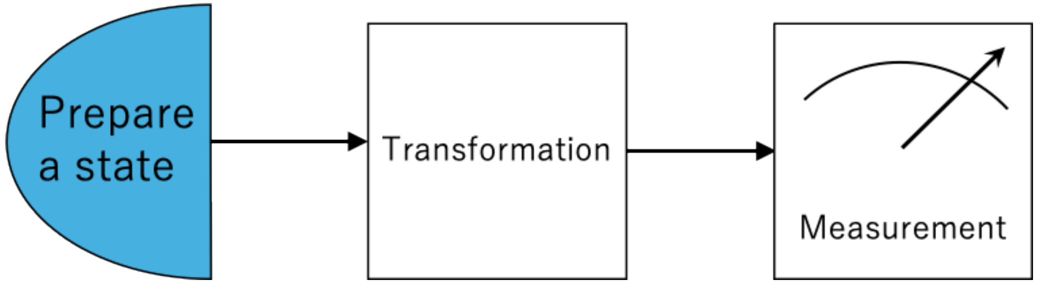
Quantum Theory

Nielsen and Chuang (2010)

Quantum theory



Quantum theory



Axiom 1. (Quantum systems and states)

Quantum systems A, B ... are associated with complex Hilbert spaces $\mathcal{H}_\mathrm{A}, \mathcal{H}_\mathrm{B}$ ... Also, a quantum state of an isolated system A is a unit vector $|\psi\rangle_\mathrm{A} \in \mathcal{H}_\mathrm{A}$

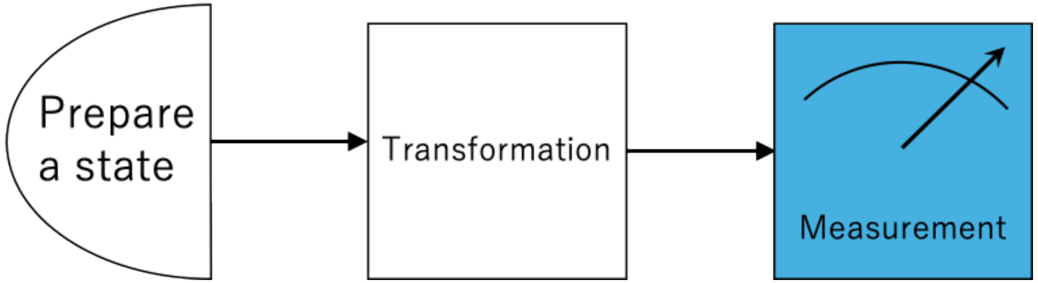
※When finite-dimensional, a Hilbert space is an inner product space.

An inner product is $\mathcal{H}_\mathrm{A} \times \mathcal{H}_\mathrm{A} \to \mathbb{C}$ like $\langle\phi|\psi\rangle_\mathrm{A}$.

If we perform $|\phi\rangle_\mathrm{B}\langle\eta|_\mathrm{A}$ to $|\psi\rangle_\mathrm{A}$, $(|\phi\rangle_\mathrm{B}\langle\eta|_\mathrm{A})|\psi\rangle_\mathrm{A} = (\langle\eta|\psi\rangle_\mathrm{A})|\phi\rangle_\mathrm{B}$

So, $|\phi\rangle_\mathrm{B}\langle\eta|_\mathrm{A}$ transforms a vector in $\mathcal{H}_\mathrm{A}$ to one in $\mathcal{H}_\mathrm{B}$ (Linear Operator)

Quantum theory



Axiom 2. (Born rule)

Observables are Hermitian operators. For an observable $X_{\mathrm{A}} \in \mathbf{Herm}(\mathcal{H}_{\mathrm{A}})$ with the eigenvalue decomposition

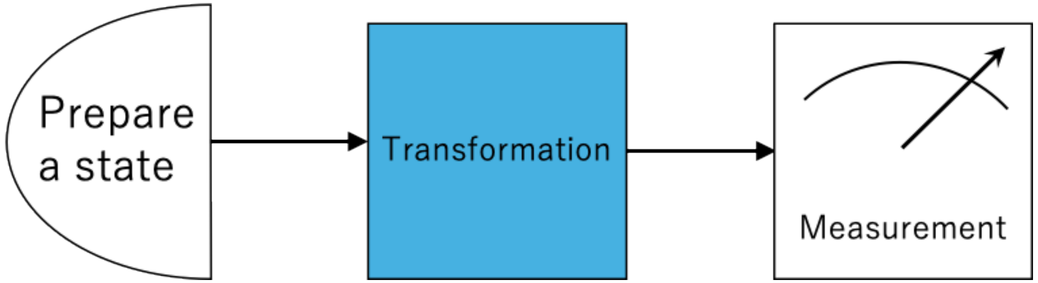
$$X_{\mathrm{A}} = \sum_{x \in \mathcal{X}} x |\phi_x\rangle\langle\phi_x|_{\mathrm{A}},$$

the probability of getting x when we perform a measurement of X_{A} to the state $|\psi\rangle_{\mathrm{A}}$ is given by $p_x = |\langle\phi_x|\psi\rangle_{\mathrm{A}}|^2$

※The expectation value is

$$\langle X_{\mathrm{A}}\rangle_{\psi} = \sum_x x p_x = \sum_x x |\langle\phi_x|\psi\rangle_{\mathrm{A}}|^2 = \langle\psi|X_{\mathrm{A}}|\psi\rangle_{\mathrm{A}} = \mathrm{Tr}[X_{\mathrm{A}}|\psi\rangle\langle\psi|_{\mathrm{A}}]$$

Quantum theory



Axiom 3. (Schrödinger equation)

A time evolution in an isolated system satisfies

$$i\hbar \frac{d}{dt} |\psi(t)\rangle_{\mathrm{A}} = H_{\mathrm{A}} |\psi(t)\rangle_{\mathrm{A}}$$

where H_{A} is a Hamiltonian, which provides the energy

※Then the time evolution is given by a unitary

$$|\psi(0)\rangle \mapsto |\psi(t)\rangle = U(t)|\psi(0)\rangle$$

where $U(t) = \exp\left(-i \frac{H_{\mathrm{A}}}{\hbar} t\right)$

Quantum theory

Axiom 4. (Composite systems)
The Hilbert space of a composite quantum system $\mathrm{A}+\mathrm{B}$ is a tensor product Hilbert space $\mathcal{H}_\mathrm{A}\otimes\mathcal{H}_\mathrm{B}$

Tensor product $\otimes: \mathcal{H}_\mathrm{A}\times\mathcal{H}_\mathrm{B}\to\mathcal{H}_\mathrm{AB}$ is an operation satisfying

$$(a|\phi_1\rangle_\mathrm{A}+b|\phi_2\rangle_\mathrm{A})\otimes|\psi\rangle_\mathrm{B}=a|\phi_1\rangle_\mathrm{A}\otimes|\psi\rangle_\mathrm{B}+b|\phi_2\rangle_\mathrm{A}\otimes|\psi\rangle_\mathrm{B}$$

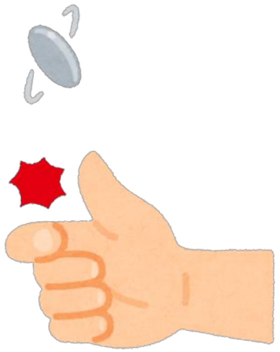
$$|\phi\rangle_\mathrm{A}\otimes(a|\psi_1\rangle_\mathrm{B}+b|\psi_2\rangle_\mathrm{B})=a|\phi\rangle_\mathrm{A}\otimes|\psi_1\rangle_\mathrm{B}+b|\phi\rangle_\mathrm{A}\otimes|\psi_2\rangle_\mathrm{B}$$

$$(\langle\eta|_\mathrm{A}\otimes\langle\zeta|_\mathrm{B})(|\phi\rangle_\mathrm{A}\otimes|\psi\rangle_\mathrm{B})=\langle\eta|\phi\rangle_\mathrm{A}\langle\zeta|\psi\rangle_\mathrm{B}$$

Pure states and mixed states

- $|\psi\rangle$: pure state

$$p_x = p|\langle\phi_x|\psi_1\rangle|^2 + (1-p)|\langle\phi_x|\psi_2\rangle|^2$$
$$= \langle\phi_x|(p|\psi_1\rangle\langle\psi_1| + (1-p)|\psi_2\rangle\langle\psi_2|)|\phi_x\rangle$$



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p → $|\psi_1\rangle$

$1-p$ → $|\psi_2\rangle$

It is reasonable to regard

$$\rho := p|\psi_1\rangle\langle\psi_1| + (1-p)|\psi_2\rangle\langle\psi_2|$$

as a state

Density Operator: $\rho \geq 0$ and $\mathrm{Tr}\rho = 1$

POVM

Measurement: get a probability distribution from ρ.

Consider $\left\{E_{\mathrm{A}}^{(k)}\right\}_{k\in\mathcal{K}}$ where $E_{\mathrm{A}}^{(k)} \geq 0$ and $\sum_{k\in\mathcal{K}} E_{\mathrm{A}}^{(k)} = I_{\mathrm{A}}$

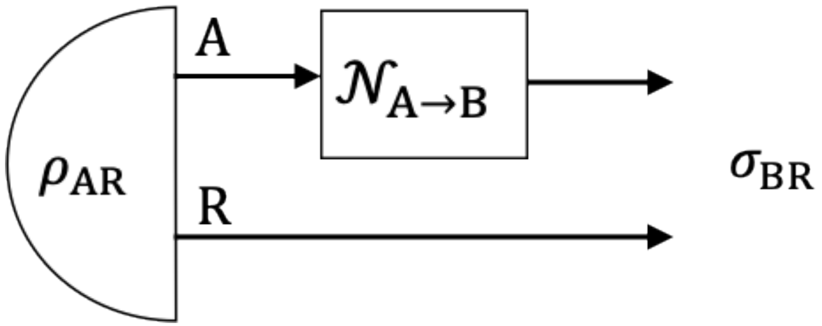
As for $p_k = \mathrm{Tr}\left[\rho_{\mathrm{A}} E_{\mathrm{A}}^{(k)}\right]$

- Since $\rho \geq 0$, $p_k \geq 0$
- Also, since $\mathrm{Tr}[\rho] = 1$,$\sum_{k\in\mathcal{K}} p_k = \sum_{k\in\mathcal{K}} \mathrm{Tr}\left[\rho_{\mathrm{A}} E_{\mathrm{A}}^{(k)}\right] = 1$

$\left\{E_{\mathrm{A}}^{(k)}\right\}_{k\in\mathcal{K}}$: Positive Operator Valued Measure (POVM)

※If all $E_{\mathrm{A}}^{(k)}$ are projection operators, Projection Valued Measure

CPTP linear map



Consider the conditions of a map $\mathcal{N}_{A\to B}$ for the state conversion

- $\sigma_B \coloneqq \mathcal{N}_{A\to B}(\rho_A)$ must be
  - $\sigma_B \geq 0$ ⇨Positive
  - $\mathrm{Tr}[\sigma_B] = 1$ ⇨Trace-Preserving

- Moreover, for a state on the systems A and any system R, ρ_{AR},

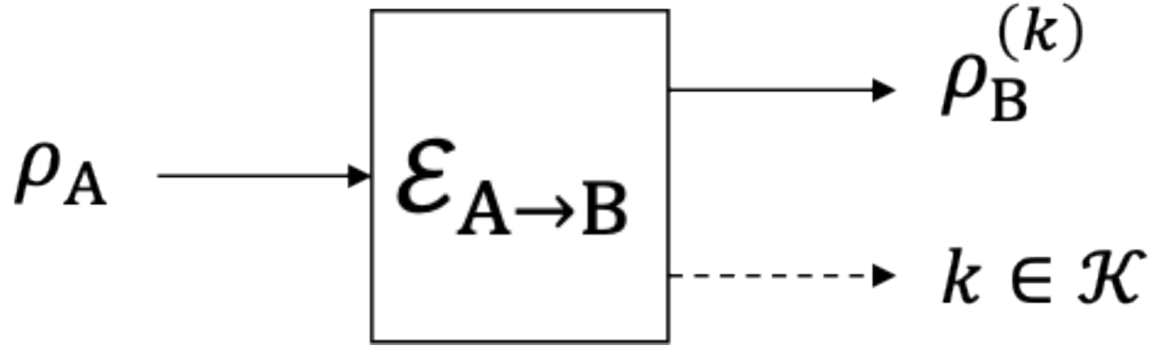
$$\sigma_{BR} \coloneqq \mathcal{N}_{A\to B} \otimes \mathrm{id}_R(\rho_{AR})$$

  - $\sigma_{BR} \geq 0$ ⇨Completely Positive
  - Of course, Trace-Preserving

Quantum measurement processes

CP-instrument

Ozawa (1984)



$\left\{\mathcal{E}_{\mathrm{A}\to\mathrm{B}}^{(k)}\right\}_{k\in\mathcal{K}}$: a family of CP trace non-increasing linear maps s.t. $\sum_{k\in\mathcal{K}} \mathcal{E}_{\mathrm{A}\to\mathrm{B}}^{(k)}$ is TP

$$\rho_{\mathrm{B}}^{(k)} = \frac{\mathcal{E}_{\mathrm{A}\to\mathrm{B}}^{(k)}(\rho_{\mathrm{A}})}{p_k}, p_k = \mathrm{Tr}[\mathcal{E}_{\mathrm{A}\to\mathrm{B}}^{(k)}(\rho_{\mathrm{A}})]$$

Specific CP-instrument

- Von Neumann's projective measurement

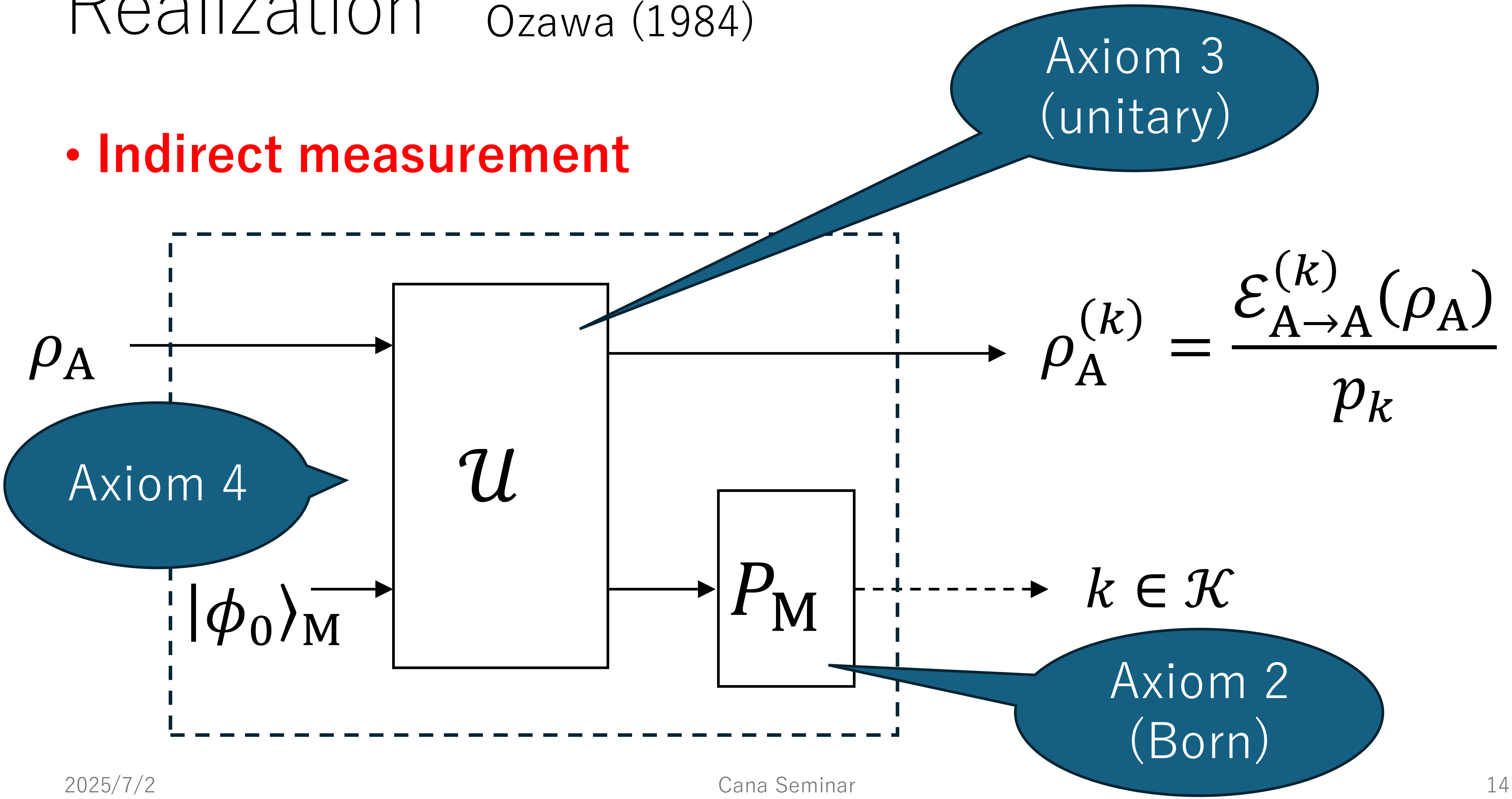
$$\{|\phi_k\rangle\langle\phi_k|_{\mathrm{A}} \cdot |\phi_k\rangle\langle\phi_k|_{\mathrm{A}}\}_{k\in\mathcal{K}}$$

For any state ρ_{A}, the (unnormalized) post-measurement state is

$$\underbrace{\langle\phi_k|\rho_{\mathrm{A}}|\phi_k\rangle}_{p_k}|\phi_k\rangle\langle\phi_k|$$

Realization Ozawa (1984)

- **Indirect measurement**



Math vs Physics

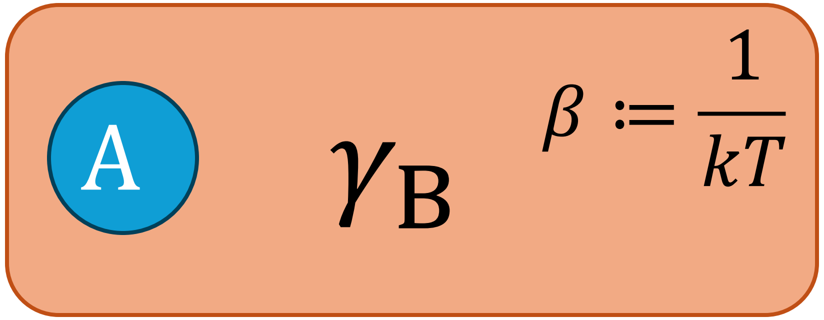
- All CP-instruments are realizable when we admit Axioms 1-4

Q. Are they also consistent with physical laws?

Axioms 1-4 do not seem to contain physical laws such as

- Conservation law (the first law of thermodynamics)
- The second law of thermodynamics
- The third law of thermodynamics...

Quantum thermodynamics



- Let H_{A} be a Hamiltonian of a system A
- $E(\rho_{\mathrm{A}}, H_{\mathrm{A}}) := \mathrm{Tr}[\rho_{\mathrm{A}} H_{\mathrm{A}}]$ is energy (expectation)
- $S(\rho_{\mathrm{A}}) := -\mathrm{Tr}[\rho_{\mathrm{A}} \ln \rho_{\mathrm{A}}]$ is von Neumann entropy
- Let H_{B} be a Hamiltonian of a bath B
- Let $\beta := \frac{1}{kT}$ be the inverse temperature
- The bath is initially in the Gibbs state $\gamma_{\mathrm{B}} := \frac{e^{-\beta H_{\mathrm{B}}}}{Z_{\mathrm{B}}}$ $\left(Z_{\mathrm{B}} := \mathrm{Tr}\left[e^{-\beta H_{\mathrm{B}}}\right]\right)$
- Free energy: $F(\rho_{\mathrm{A}}, H_{\mathrm{A}}, \beta) = E(\rho_{\mathrm{A}}, H_{\mathrm{A}}) - \beta^{-1} S(\rho_{\mathrm{A}})$

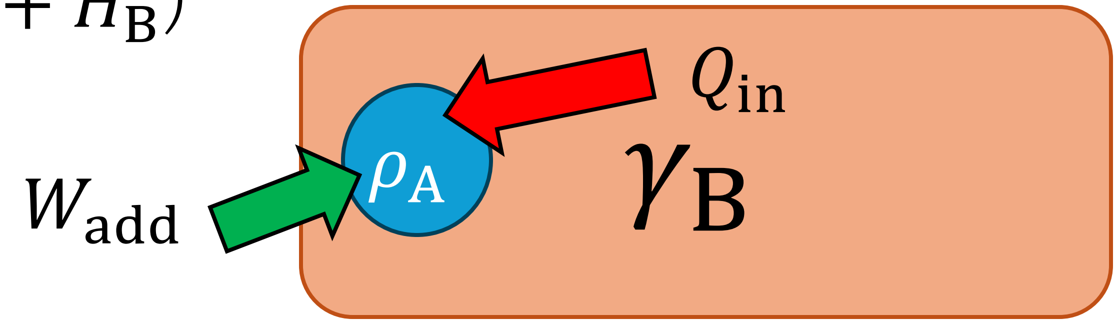
Esposito and Van den Broek (2011)

Work in quantum thermodynamics

- Work of the transition $\rho_A \otimes \gamma_B \to \sigma_{AB}$ is as follows:

$$W_{\mathrm{add}} = \Delta E_{\mathrm{AB}} = E(\sigma_{\mathrm{AB}}, H'_{\mathrm{AB}}) - E(\rho_{\mathrm{A}} \otimes \gamma_{\mathrm{B}}, H_{\mathrm{AB}})$$

(☆ $H_{\mathrm{AB}} = H_{\mathrm{A}} + H_{\mathrm{B}}$ and $H'_{\mathrm{AB}} = H'_{\mathrm{A}} + H_{\mathrm{B}}$)



※Why this works?

☆ $\to W_{\mathrm{add}} = \Delta E_{\mathrm{AB}} = \Delta E_{\mathrm{A}} + \Delta E_{\mathrm{B}}$

Let us define $Q_{\mathrm{in}} \coloneqq -\Delta E_{\mathrm{B}}$ (heat absorbed in the system)

Then we have the 1st law of thermodynamics

$$\Delta E_{\mathrm{A}} = W_{\mathrm{add}} + Q_{\mathrm{in}}$$

The second law of thermodynamics

【The (nonequilibrium) 2nd law】

$$W_{\mathrm{add}} \geq \Delta F_{\mathrm{A}}$$

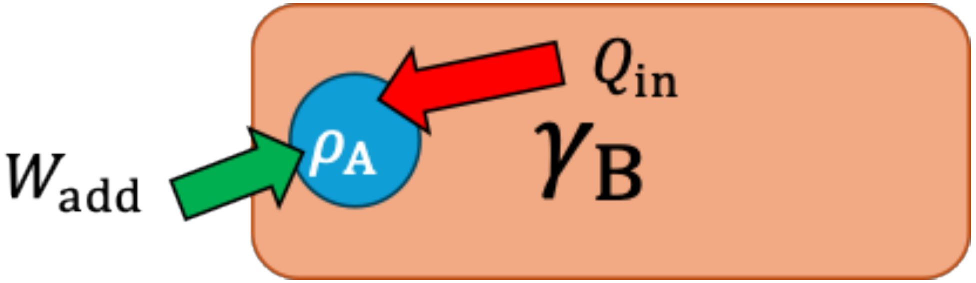
$$\Delta F_{\mathrm{A}} := F(\sigma_{\mathrm{A}}, H'_{\mathrm{A}}, \beta) - F(\rho_{\mathrm{A}}, H_{\mathrm{A}}, \beta)$$

Esposito and Van den Broek (2011)

Notice that

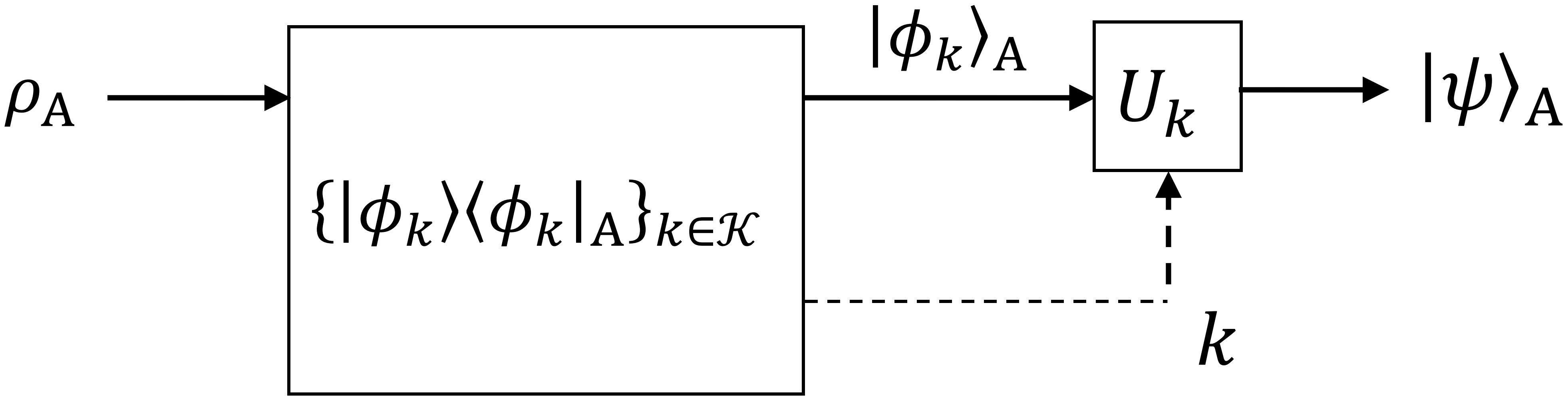
$$W_{\mathrm{add}} = \Delta E_{\mathrm{A}} - Q_{\mathrm{in}} = \Delta F_{\mathrm{A}} + \beta^{-1}\Delta S_{\mathrm{A}} - Q_{\mathrm{in}}$$

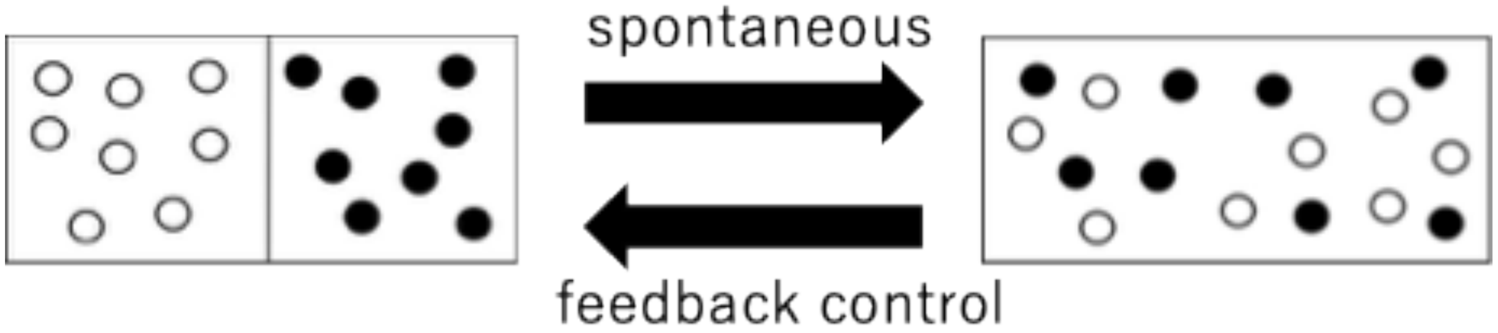
So, $W_{\mathrm{add}} \geq \Delta F_{\mathrm{A}} \Leftrightarrow \Sigma := \Delta S_{\mathrm{A}} - \beta Q_{\mathrm{in}} \geq 0$



When adiabatic ($Q_{\mathrm{in}} = 0$), $\Delta S_{\mathrm{A}} \geq 0$

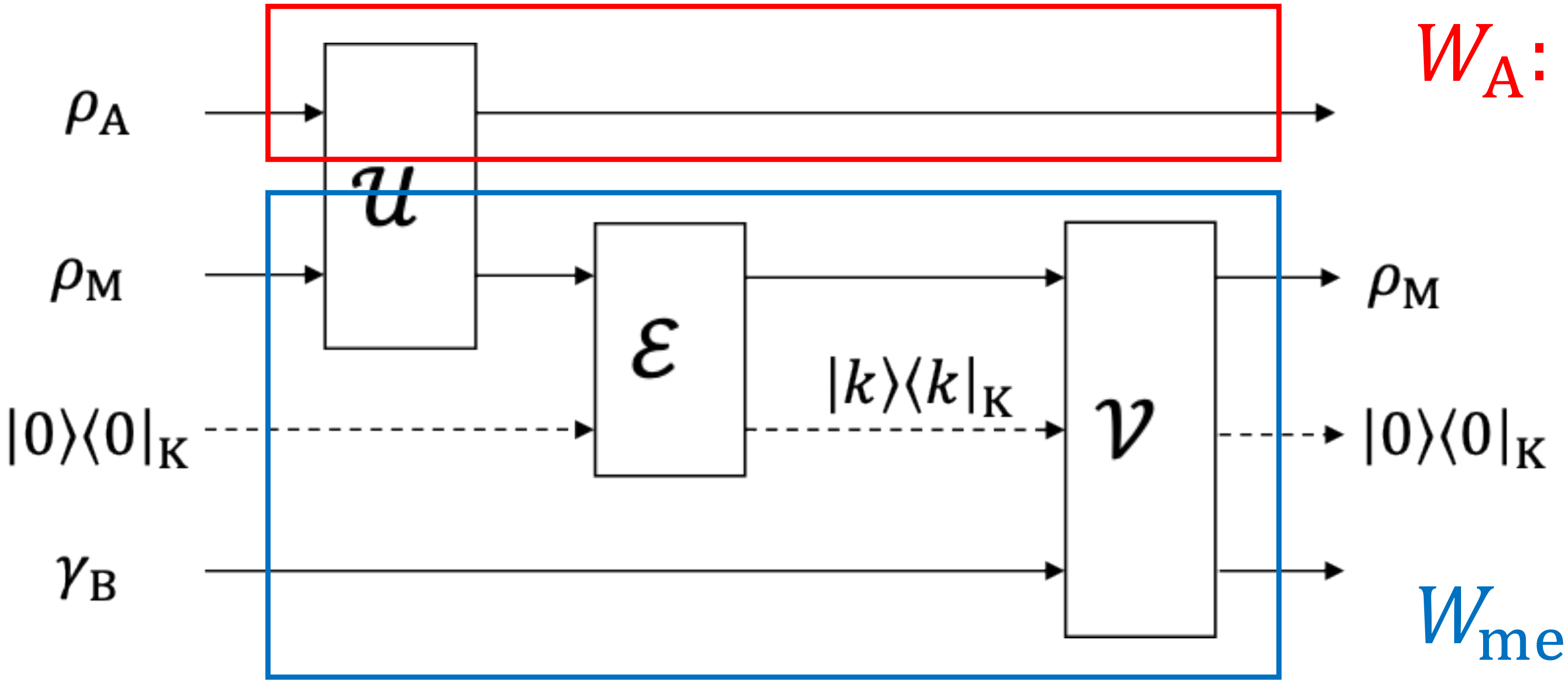
Q. Are all CP instruments consistent with the 2nd law?





Maxwell (1871)

Q. Are all indirect measurements consistent with the 2nd law?



$W_A := \Delta E_A$

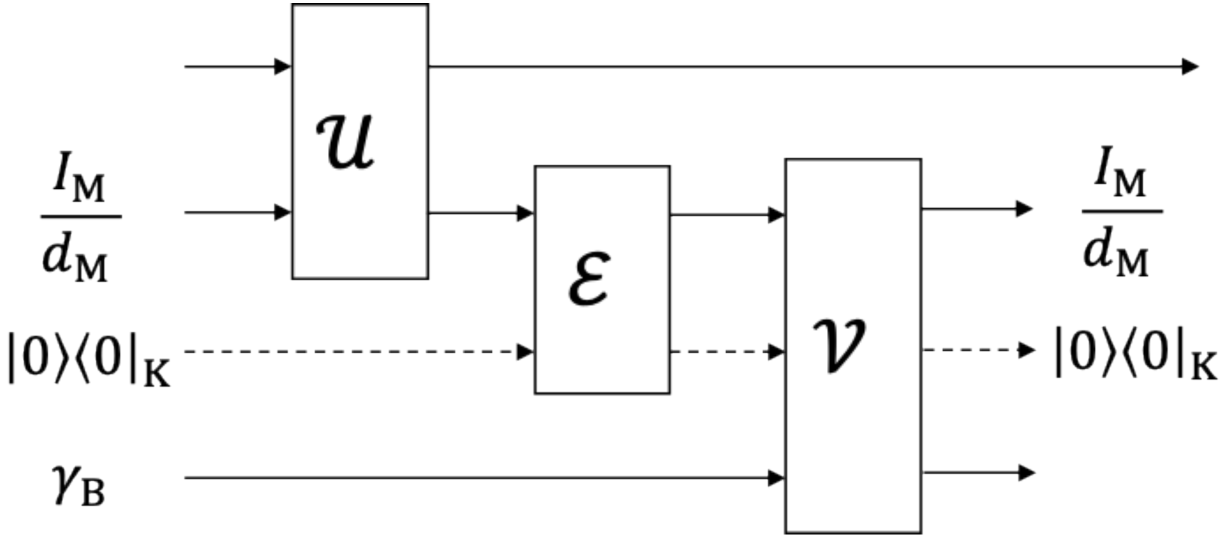
By resetting M and K, we can fairly compare with the previous slide

$W_{meas} := \Delta E_{MKB}$

Sagawa and Ueda (2008, 2009), Funo et al. (2013), Abdelkhalek et al. (2016) said

Even if $W_A < \Delta F_A$ (violation of the 2nd law on A), $W_A + W_{meas} \geq \Delta F_A$!

Q. Are all indirect measurements consistent with the 2nd law?



Indirect measurement:

$$\rho_A \otimes \frac{I_M}{d_M} \otimes |0\rangle\langle 0|_K$$

$$\mapsto \rho_A \otimes |\psi\rangle\langle\psi|_M \otimes \sum_k p_k |k\rangle\langle k|_K$$

$$W_A = 0, \Delta F_A = 0$$
$$W_{meas} = \beta^{-1}[H(\{p_k\}) - \ln d_M]$$

➡ If $|\mathcal{K}| < d_M$, $W_A + W_{meas} < \Delta F_A$

Assumptions in previous works

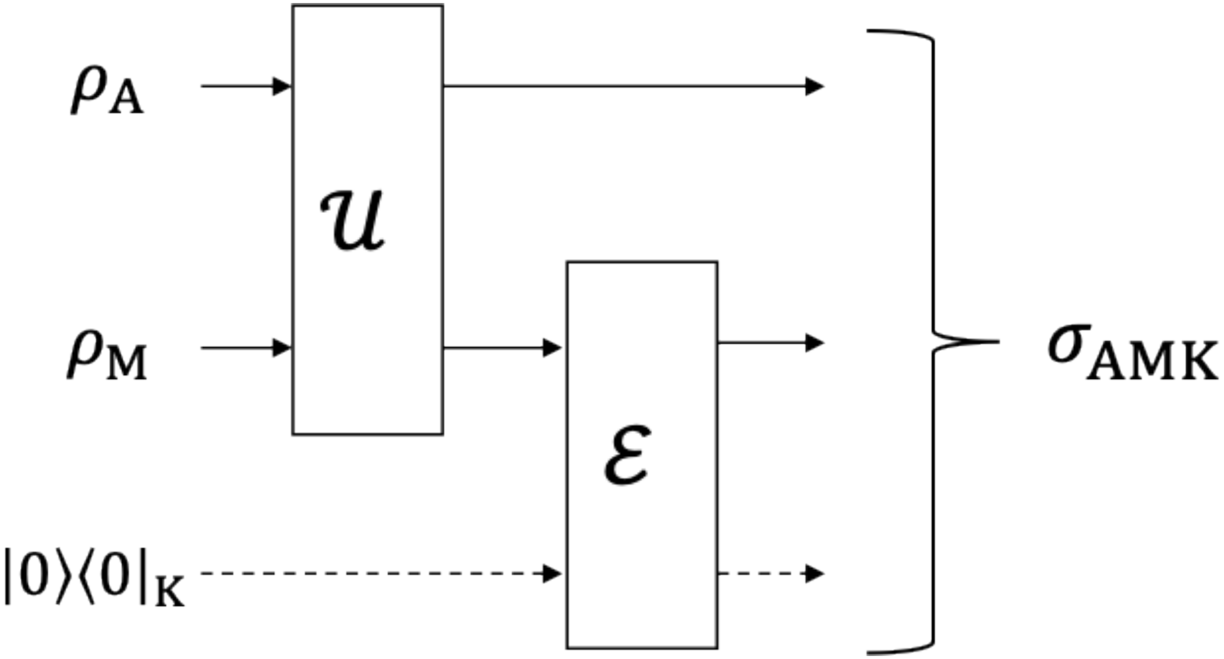
➢ Sagawa and Ueda imposed several assumptions

① Efficient instrument
② Projective measurement (Luders instrument)
③ Post-measurement state is in a product state
④ Initial state of the target system is in the thermal state
⑤ Feedback control is the pure unitary on the target system
⑥ Memory is in a thermal state initially and before the erasure

Problem 1: Ultimately, which assumption makes a measurement consistent with the second law?
Problem 2: Can all assumptions coexist?

What is the condition for the measurement to be consistent with the 2nd law?

Minagawa et al. (2025)



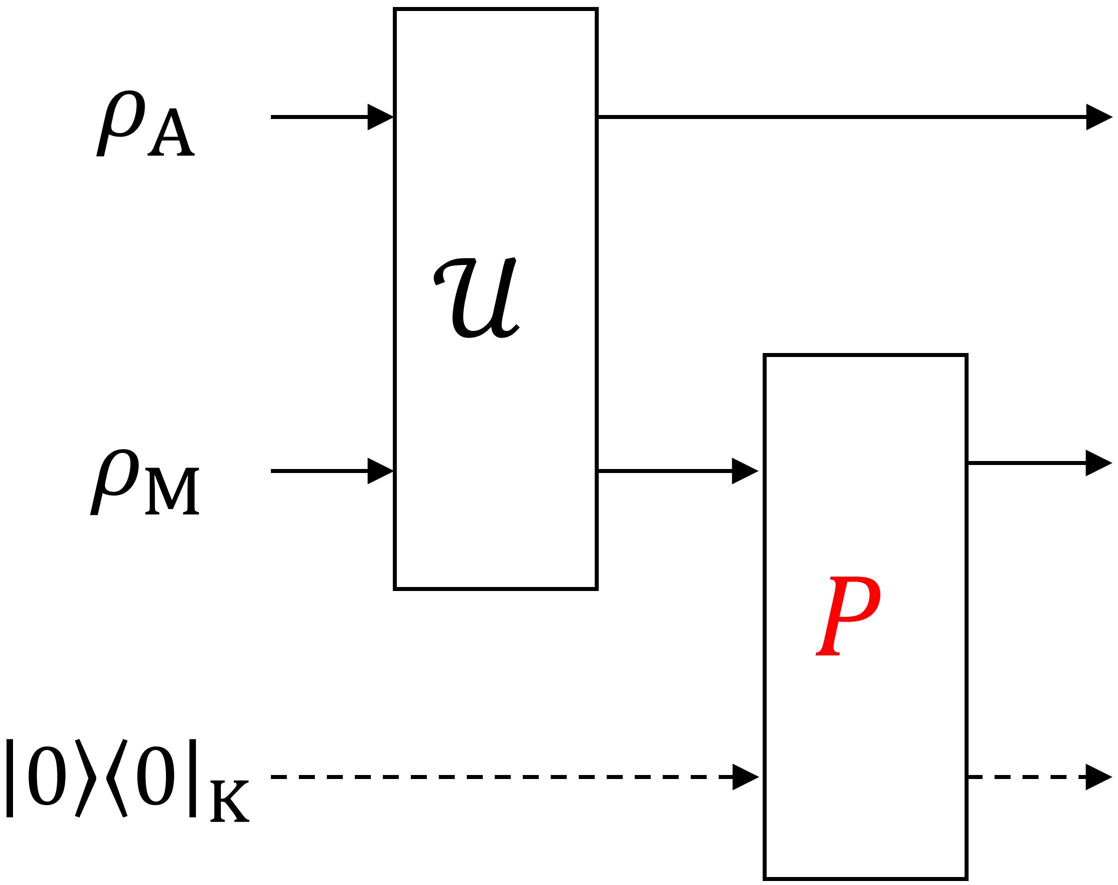
$$\Delta S_{\mathrm{AMK}} \geq -I(\mathrm{A{:}\,M}|\mathrm{K})_\sigma$$

$$W_{\mathrm{A}} + W_{\mathrm{meas}} \geq \Delta F_{\mathrm{A}}$$

※$I(\mathrm{A{:}\,M}|\mathrm{K})_\sigma = S(\mathrm{A}|\mathrm{K})_\sigma + S(\mathrm{M}|\mathrm{K})_\sigma - S(\mathrm{AM}|\mathrm{K})_\sigma \geq 0$

See e.g., Wilde (2017)

Comparison with previous works



Sagawa and Ueda (2008, 2009),
Funo et al. (2013),
Abdelkhalek et al. (2016)

Projective measurements do not decrease the entropy

Nielsen and Chuang (2010)

$$\Delta S_{\mathrm{AMK}} \geq 0 \geq -I(\mathrm{A{:}\,M}|\mathrm{K})_{\sigma}$$

Conclusion

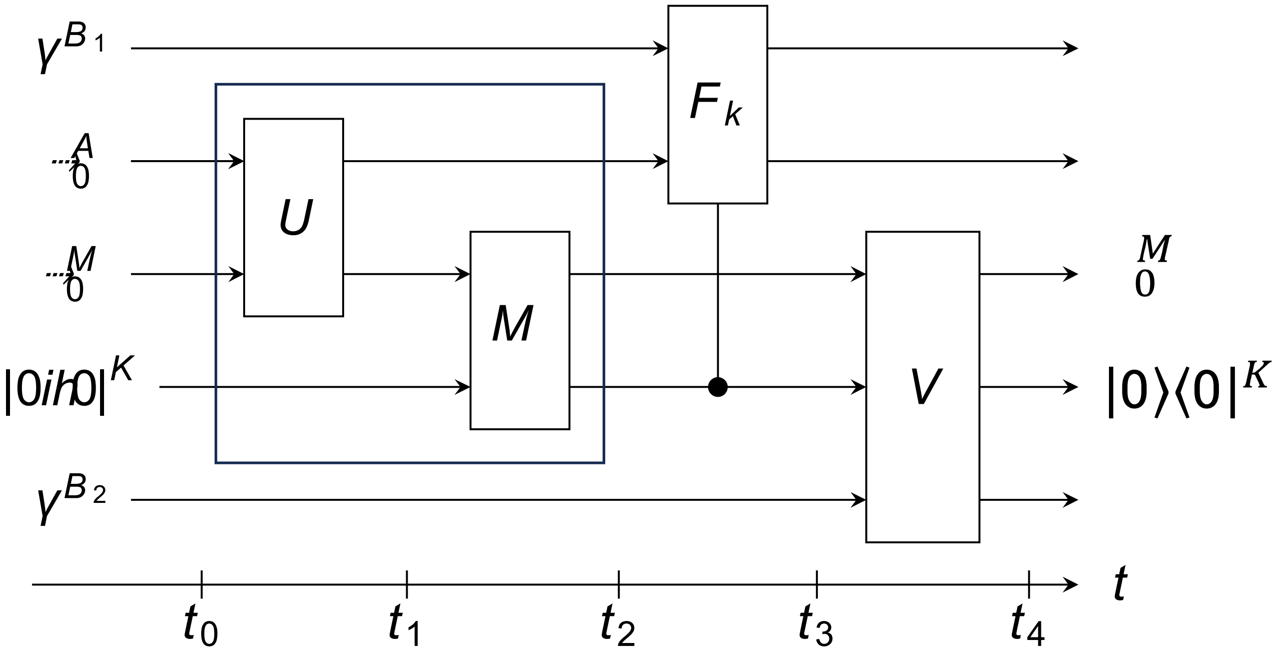
- Targetting all CP-instruments, we consider the consistency between quantum measurement processes and the 2nd law

- We found an information-theoretic inequality for the CP-instrument to be consistent with the 2^{nd} law

$$\Delta S_{\mathrm{AMK}} \geq -I(A{:}M|K)_{\sigma}$$

- Once we assume the previous works' assumptions, we can immediately see that they recover the 2^{nd} law

Appendix: General work formulas



Minagawa et al. (2025)

Work definition

$$W_{\mathrm{A}} := -\Delta E^{A}_{0\to 2} - \Delta E^{B_1 A}_{2\to 3}$$

$$W_{\mathrm{meas}} := \Delta E^{MK}_{0\to 2} + \Delta E^{MKB_2}_{3\to 4}$$

Groenewold—Ozawa information gain (Groenewold 1971, Ozawa 1986)

$$I_{\mathrm{GO}} := S(A)_{\rho_0} - S(A|K)_{\rho_2} \gtreqless 0$$

$$W_{\mathrm{A}} = -\Delta F^{A}_{0\to 4} + \beta^{-1}[I_{\mathrm{GO}} - (I(A:K)_{\rho_3} + S^{B_1}_{\mathrm{irr}})]$$

$$W_{\mathrm{meas}} = \beta^{-1}[I_{\mathrm{GO}} + \Delta S^{AMK}_{0\to 2} + I(A:M|K)_{\rho_2} + S^{B_2}_{\mathrm{irr}}]$$

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