

Causal Influence vs signaling for interacting quantum channels

Kathleen Barse, Paolo Perinotti and Alessandro Tosini,
Leonardo Vaglini

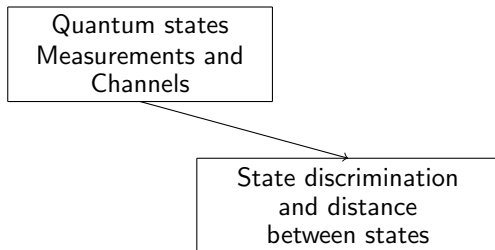
University of Aix-Marseille

23/09/2025

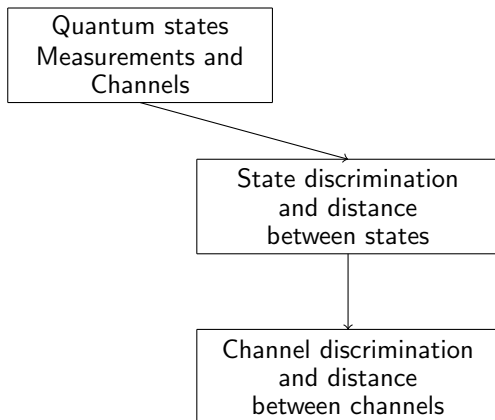
Outline

Quantum states
Measurements and
Channels

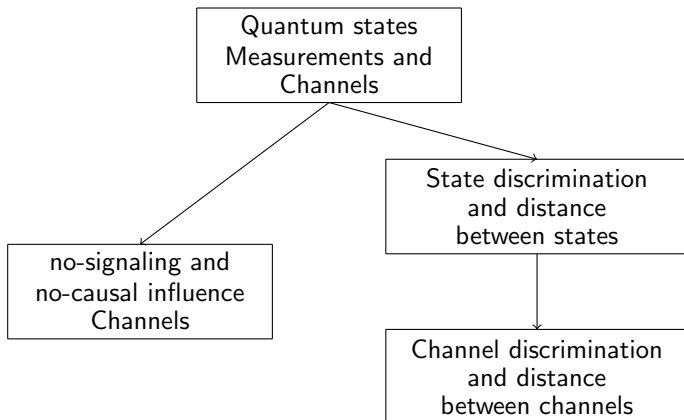
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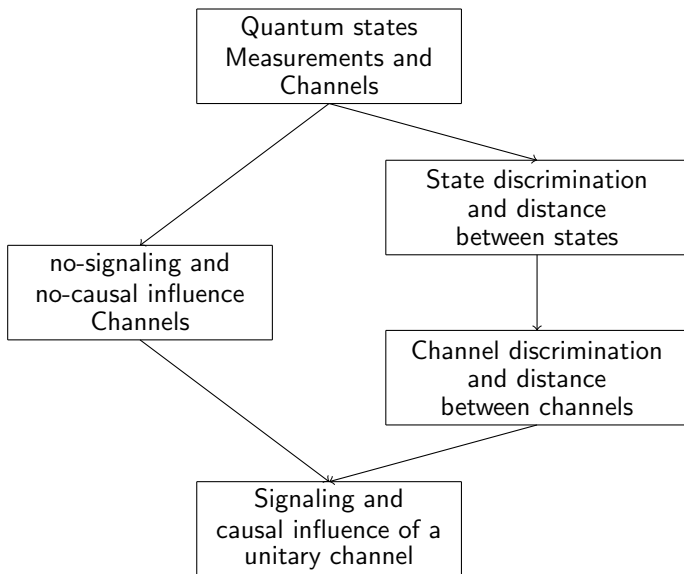
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- ▶ **positivity:** $\rho \geq 0$ (which means $\langle \psi | \rho | \psi \rangle \geq 0 \quad \forall |\psi\rangle$);
- ▶ **normalization:** $\text{Tr}(\rho) = 1$.

any quantum state is an operator satisfying such properties

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Reversible channels: $\mathcal{U} : \mathcal{D}(\mathcal{H}) \rightarrow \mathcal{D}(\mathcal{H})$ s.t. $\mathcal{U}\mathcal{U}^{-1} = \mathcal{U}^{-1}\mathcal{U} = \mathcal{I}$

$$\rho \rightarrow U\rho U^\dagger, \quad UU^\dagger = U^\dagger U = I$$

Diagrammatic notation

States and measurements

$\overbrace{\quad}^A$
systems

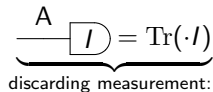
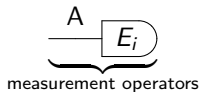
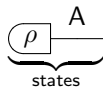
$\underbrace{\overbrace{\quad}^A}_{\rho}$
states

$\underbrace{\overbrace{\quad}^A}_{E_i}$
measurement operators

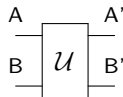
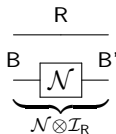
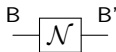
$\underbrace{\overbrace{\quad}^A}_I = \text{Tr}(\cdot I)$
discarding measurement:

Diagrammatic notation

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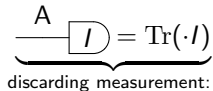
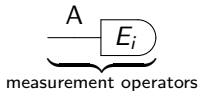
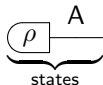


Channels:

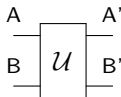
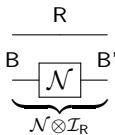


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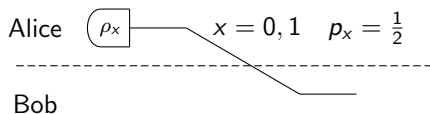
Combining states, measurements and channels

$$P(i|\mathbf{E}, \rho) =: \text{Diagram of } \rho \text{ on } A \text{ followed by measurement } E_i$$

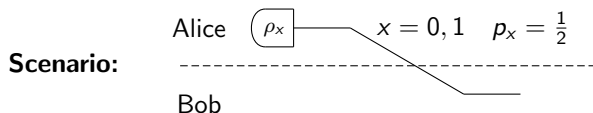
$$P(i|\mathbf{E}, \mathcal{N}, \rho) =: \text{Diagram of } \rho \text{ on } B \text{ followed by channel } \mathcal{N} \text{ and measurement } E_i$$

Operational distance for quantum states I

Scenario:



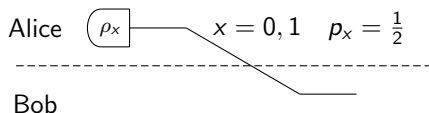
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Task: Bob has to guess which state, ρ_0 or ρ_1 , has been prepared.

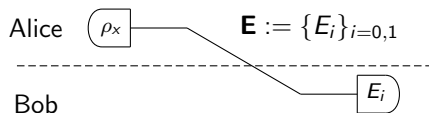
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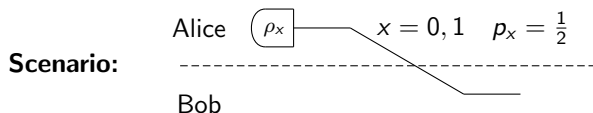


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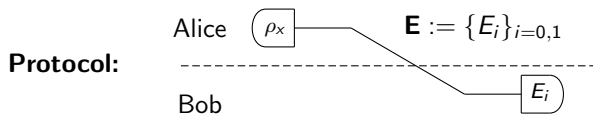
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He declares that the state is

- ▶ ρ_0 if the outcome is 0;
- ▶ ρ_1 if the outcome is 1.

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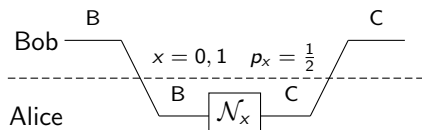
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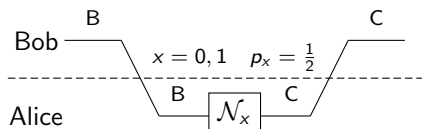
Channel distance: the Diamond norm

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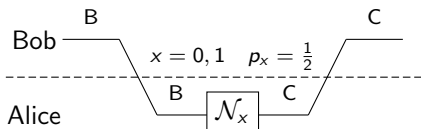
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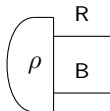
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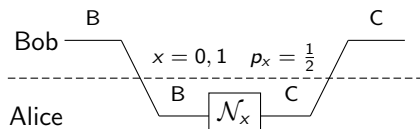
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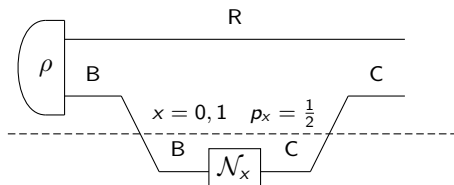
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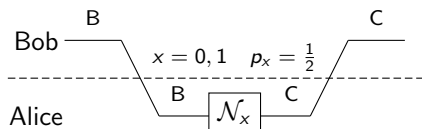
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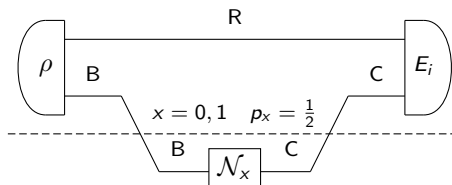
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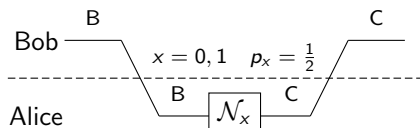
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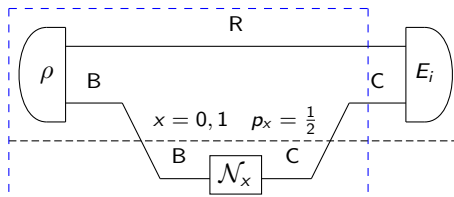
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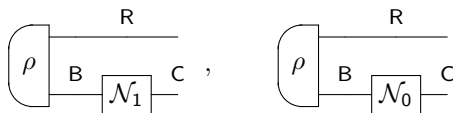
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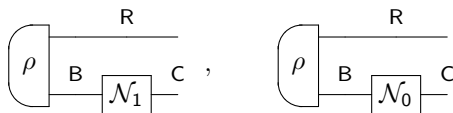
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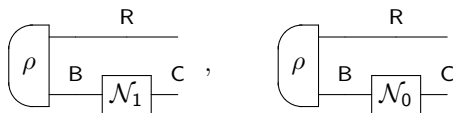
The success probability is given by

$$p_{\text{succ}}^{\text{max}} = \frac{1}{2} \left(1 + \frac{1}{2} \left\| \left(\begin{array}{c} \text{Diagram of } \mathcal{N}_1 \\ \text{Diagram of } \mathcal{N}_0 \end{array} \right) \right\| \right)$$

The diagram inside the norm is the difference of the two state discrimination tasks shown above, enclosed in a double vertical line.

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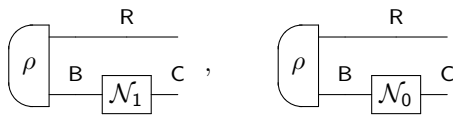


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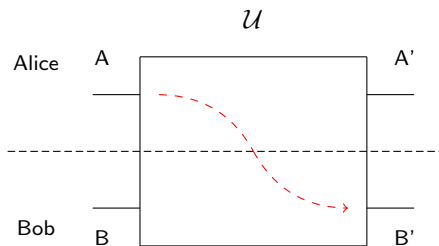
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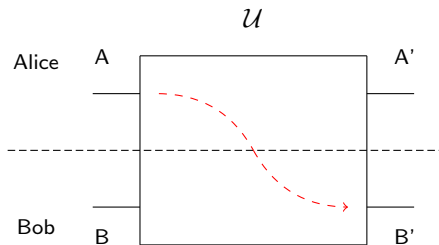
The diamond distance is defined as

$$\|\mathcal{N}_1 - \mathcal{N}_2\|_{\diamond} := \sup_R \max_{\rho} \left\| \begin{array}{c} R \\ \rho \text{---} B \text{---} \mathcal{N}_1 \text{---} C \end{array} - \begin{array}{c} R \\ \rho \text{---} B \text{---} \mathcal{N}_0 \text{---} C \end{array} \right\|$$

Introduction and no-signaling channels

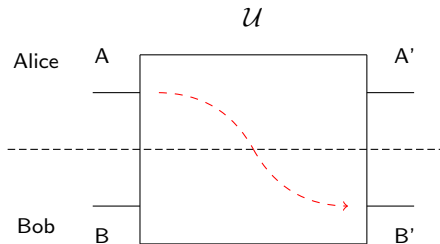


Introduction and no-signaling channels



Signaling from A to B': By varying her local state, Alice can modify the outcome probabilities of a Bob's local measurement

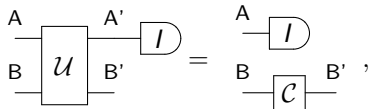
Introduction and no-signaling channels



Signaling from A to B': By varying her local state, Alice can modify the outcome probabilities of a Bob's local measurement

[Schumacher, Westmoreland QIP 4, 05]

no-signaling from A to B' if:

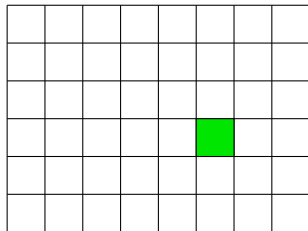


No-causal influence channels

For each site $\square = \mathcal{H}_\ell \leftrightarrow \mathcal{A}_\ell$

Cellular Automata:

$$U : \bigotimes_\ell \mathcal{H}_\ell \rightarrow \bigotimes_\ell \mathcal{H}_\ell.$$



No-causal influence channels

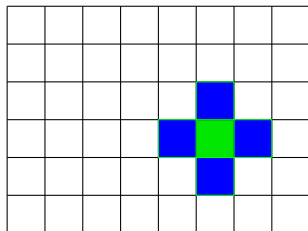
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$$U^{-1} A_{\ell_0} U = A_{N(\ell)}$$



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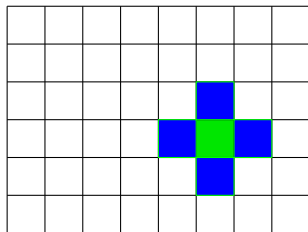
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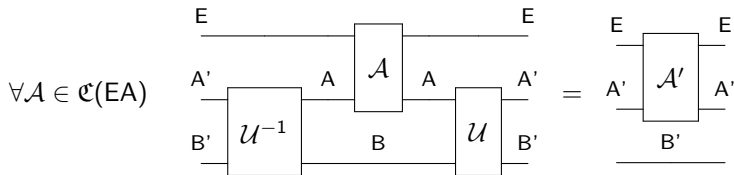
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[Perinotti, Quantum 4, 20, Perinotti, Quantum 5, 21]

No-causal influence from A to B'



No-causal influence gate

[Perinotti, Quantum 4, 20, Perinotti, Quantum 5, 21]

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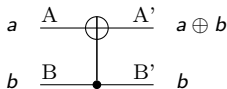
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- ▶ In general: no-causal influence $\implies$ no-signalling
- ▶ In Quantum Theory (QT) no-causal influence $\iff$ no-signaling
- ▶ In general the viceversa is not true: no-signaling $\nRightarrow$ no-causal influence

Counterexample: classical information theory

$\text{CNOT}(a, b) = (a \oplus b, b)$ for $a, b \in \{0, 1\}$



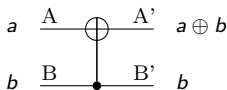
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- ▶ In general: no-causal influence $\implies$ no-signalling
- ▶ In Quantum Theory (QT) no-causal influence $\iff$ no-signaling
- ▶ In general the viceversa is not true: no-signaling $\nRightarrow$ no-causal influence

Counterexample: classical information theory

$\text{CNOT}(a, b) = (a \oplus b, b)$ for $a, b \in \{0, 1\}$



It is no signalling from the target to the control

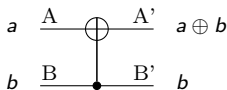
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It allows the creation of correlations between Alice and Bob at the output depending on the Alice's input.

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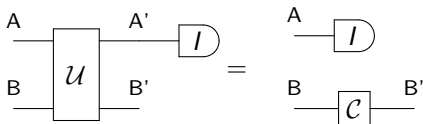
2nd question: is there a continuity relation between such quantifiers?

3rd question: is there a remnant of their inequivalence in QT?

Definition of signalling

[Barse, Perinotti, Tosini, LV, PRR 6, 043305, 24]

Negate the no-signalling condition

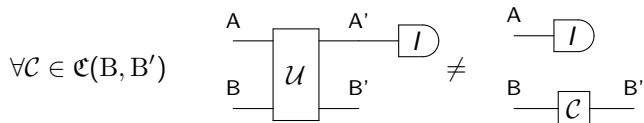
$$\exists \mathcal{C} \in \mathfrak{C}(B, B') \text{ s.t.}$$


The diagrammatic equation shows two circuit expressions separated by an equals sign. The left expression consists of a rectangular box labeled $\mathcal{U}$ with two input wires on the left labeled A and B , and two output wires on the right labeled A' and B' . The output wire A' is connected to a rounded rectangular box labeled I . The right expression consists of two separate components: a rounded rectangular box labeled I with a single input wire labeled A , and a square box labeled $\mathcal{C}$ with two input wires labeled B and B' .

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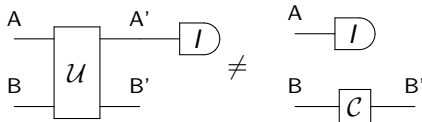


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$$\forall C \in \mathfrak{C}(B, B')$$



How different are they?

Definition of signalling

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$$\forall C \in \mathfrak{C}(B, B') \quad \begin{array}{c} A \\ \hline \boxed{U} \\ B \end{array} \begin{array}{c} A' \\ \hline \boxed{I} \\ B' \end{array} \neq \begin{array}{c} A \\ \hline \boxed{I} \\ B \end{array} \begin{array}{c} B' \\ \hline \boxed{C} \end{array}$$

How different are they?

$$\left\| \begin{array}{c} A \\ \hline \boxed{U} \\ B \end{array} \begin{array}{c} A' \\ \hline \boxed{I} \\ B' \end{array} - \begin{array}{c} A \\ \hline \boxed{I} \\ B \end{array} \begin{array}{c} B' \\ \hline \boxed{C} \end{array} \right\|_{\diamond}$$

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$$\inf_{C \in \mathfrak{C}(B, B')} \left\| \begin{array}{c} A \\ \hline \boxed{U} \\ B \end{array} \begin{array}{c} A' \\ \hline \boxed{I} \\ B' \end{array} - \begin{array}{c} A \\ \hline \boxed{I} \\ B \end{array} \begin{array}{c} B' \\ \hline \boxed{C} \end{array} \right\|_{\diamond}$$

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How different are they?

$$\Sigma(U) := \inf_{C \in \mathfrak{C}(B, B')} \left\| \begin{array}{c} A \\ \hline \boxed{U} \\ B \end{array} \begin{array}{c} A' \\ \hline \boxed{I} \\ B' \end{array} - \begin{array}{c} A \\ \hline \boxed{I} \\ B \end{array} \begin{array}{c} B' \\ \hline \boxed{C} \end{array} \right\|_{\diamond}$$

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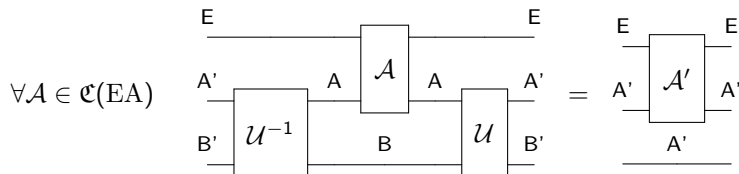
$$\Sigma(\mathcal{U}) := \inf_{C \in \mathfrak{C}(B, B')} \left\| \begin{array}{c} A \\ \hline \boxed{\mathcal{U}} \\ \hline B \end{array} \begin{array}{c} A' \\ \hline \boxed{I} \\ \hline B' \end{array} - \begin{array}{c} A \\ \hline \boxed{I} \\ \hline B \end{array} \begin{array}{c} B' \\ \hline \boxed{C} \\ \hline \end{array} \right\|_{\diamond}$$

$\mathcal{U}$ is no-signalling $\iff \Sigma(\mathcal{U}) = 0$;

$\Sigma(\mathcal{U}) \leq 2$

Equivalent condition for no causal influence

The actual definition of a no-causal influence channel is "useless"



Equivalent condition for no causal influence

The actual definition of a no-causal influence channel is "useless"

$$\forall \mathcal{A} \in \mathfrak{C}(EA)$$

But it is sufficient to check the condition for the swap operator only:

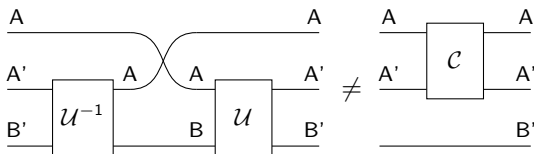
$$\exists \mathcal{C} \in \mathfrak{C}(EA', EA')$$

[Perinotti, Quantum 5, 21]

Causal influence of a unitary channel

Negate the alternative condition for no-causal influence

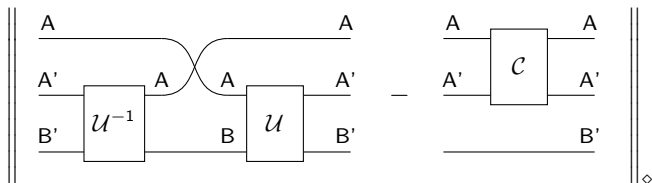
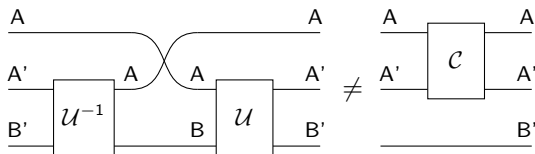
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Causal influence of a unitary channel

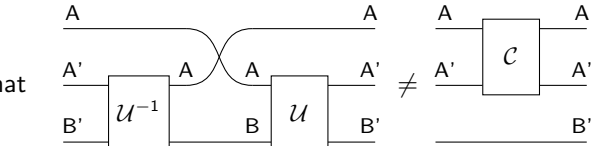
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Causal influence of a unitary channel

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$$\forall \mathcal{C} \in \mathfrak{C}(AA') \text{ such that}$$


$$\inf_{\mathcal{C} \in \mathfrak{C}(AA')} \left\| \begin{array}{c} \text{Diagram of } U^{-1} \text{ and } U \text{ with crossing} \\ \text{Diagram of } C \end{array} \right\|$$

Causal influence of a unitary channel

Negate the alternative condition for no-causal influence

$\forall \mathcal{C} \in \mathfrak{C}(AA')$ such that

$C(\mathcal{U}) := \inf_{\mathcal{C} \in \mathfrak{C}(AA')} \left\| \begin{array}{c} \text{Circuit with } \mathcal{U}^{-1} \text{ and } \mathcal{U} \\ \text{Circuit with } \mathcal{C} \end{array} \right\|$

Causal influence of a unitary channel

Negate the alternative condition for no-causal influence

$$\forall \mathcal{C} \in \mathfrak{C}(AA') \text{ such that}$$

The diagram shows two circuit expressions separated by an inequality symbol $\neq$. The left expression is a unitary $\mathcal{U}$ with three input lines: A , A' , and B' , and three output lines: A , A' , and B' . A swap gate is applied to the A and A' lines before the unitary. The right expression is a channel $\mathcal{C}$ with two input lines A and A' , and two output lines A and A' , plus a third output line B' that is not connected to any input.

$$C(\mathcal{U}) := \inf_{\mathcal{C} \in \mathfrak{C}(AA')} \left\| \begin{array}{c} \text{Diagram of } \mathcal{U} \text{ with swap} \\ \text{Diagram of } \mathcal{C} \end{array} \right\|$$

The diagram shows the definition of $C(\mathcal{U})$ as the infimum over all channels $\mathcal{C} \in \mathfrak{C}(AA')$ of the norm of the difference between two circuit diagrams. The first diagram is the unitary $\mathcal{U}$ with a swap on the A and A' lines. The second diagram is the channel $\mathcal{C}$ with inputs A and A' and outputs A and A' , and a third output line B' .

$\mathcal{U}$ does not induce causal influence from A to B' $\iff C(\mathcal{U}) = 0$;

$C(\mathcal{U}) \leq 2$.

2nd question: Is there a continuity relation between Σ and C ?

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YES

Continuity relation between Σ and C

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1. if $C(\mathcal{U}) \leq \varepsilon$ then $\Sigma(\mathcal{U}) \leq ?$

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small causal influence $\implies$ small signalling

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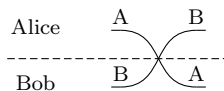
2. if $\Sigma(\mathcal{U}) \leq \varepsilon$ then $C(\mathcal{U}) \leq ?$

$$C(\mathcal{U}) \leq 2\sqrt{2}\Sigma(\mathcal{U})^{\frac{1}{2}} \leq 2\sqrt{2\varepsilon}$$

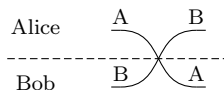
small signaling $\implies$ small causal influence.

3rd question: Can we spot the
inequivalence between signalling and causal
influence even in quantum theory?

Examples: Swap and CNOT

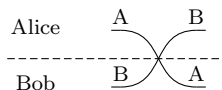


Examples: Swap and CNOT



For the qubit case $d_A = d_B = 2$: $C(\mathcal{S}) = 2$, $\Sigma(\mathcal{S}) = \frac{3}{2}$ which is not the maximum value that can be achieved, in principle, but...

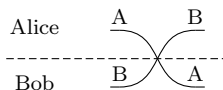
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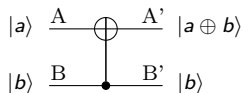
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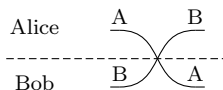
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$\text{CNOT}(|a\rangle \otimes |b\rangle) = |a \oplus b\rangle \otimes |b\rangle$ for all basis vector $|a\rangle \otimes |b\rangle$



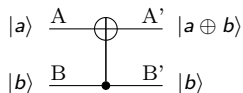
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$$C(\text{CNOT}) = 2, \Sigma(\text{CNOT}) = 1$$

Recap

[Barsse, Perinotti, Tosini, LV, arXiv:2505.14120, 25]

	CNOT	Swap $d_A = 2$	Swap	CNOT $^{\otimes n}$
$\Sigma(\mathcal{U})$	1	$\frac{3}{2}$	$2^{\frac{d_A^2-1}{d_A^2}}$	$2^{\frac{2^n-1}{2^n}}$
$C(\mathcal{U})$	2	2	2	2

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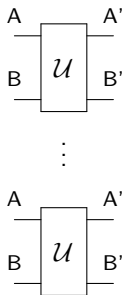
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- ▶ The Swap achieves the maximum value for both signalling and causal influence;
- ▶ The CNOT achieves the maximum value for the causal influence, but not for the signalling.

Beyond the single-shot scenario

[Barsse, Perinotti, Tosini, LV, arXiv:2505.14120, 25]



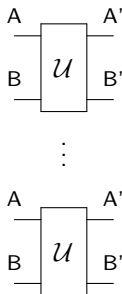
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For all unitary channels $\mathcal{U}$ and $\mathcal{V}$,

$$\Sigma(\mathcal{U} \otimes \mathcal{V}) \geq \max\{\Sigma(\mathcal{U}), \Sigma(\mathcal{V})\}$$

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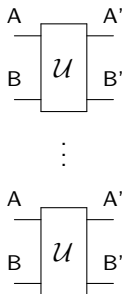
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$\Sigma(\mathcal{U}^{\otimes n})$ and $C(\mathcal{U}^{\otimes n})$ converge:

$$\Sigma_{\infty}(\mathcal{U}) := \lim_{n \rightarrow +\infty} \Sigma(\mathcal{U}^{\otimes n})$$

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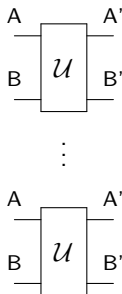
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$$\Sigma_{\infty}(\mathcal{S}) = \Sigma_{\infty}(\mathcal{C}_X) = 2$$



Conclusions

1. We have introduced two quantifiers to assess the amount of signalling and causal influence induced by a unitary channel;
2. we have shown that (in QT) if a unitary allows a small amount of signaling than it produces a small amount of causal influence (and viceversa);
3. We have computed such quantities in two explicit cases (Swap and CNOT);
4. We have studied the monotonicity of Σ and C under tensor product.