Causal Influence vs signaling for interacting quantum channels

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Leonardo Vaglini

University of Aix-Marseille

23/09/2025

Outline

Quantum states
Measurements and
Channels

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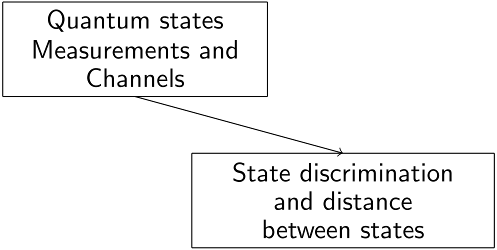
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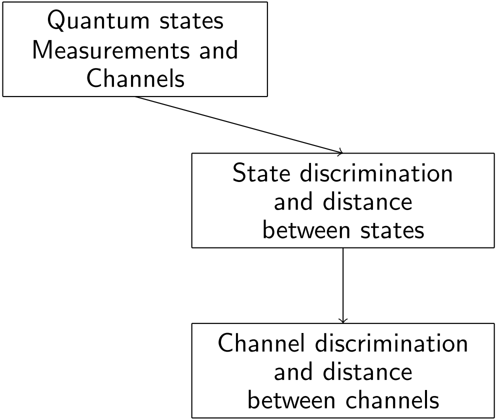
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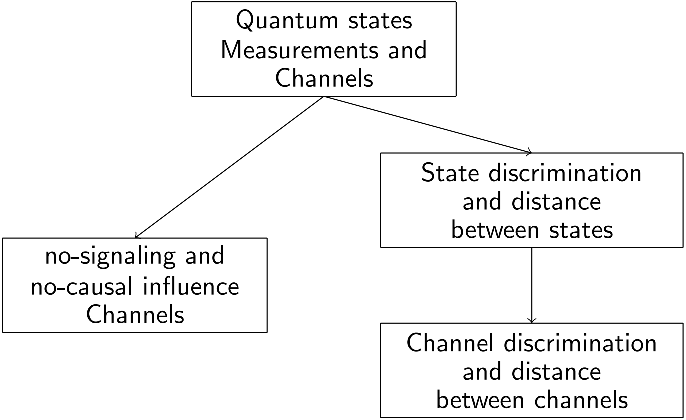
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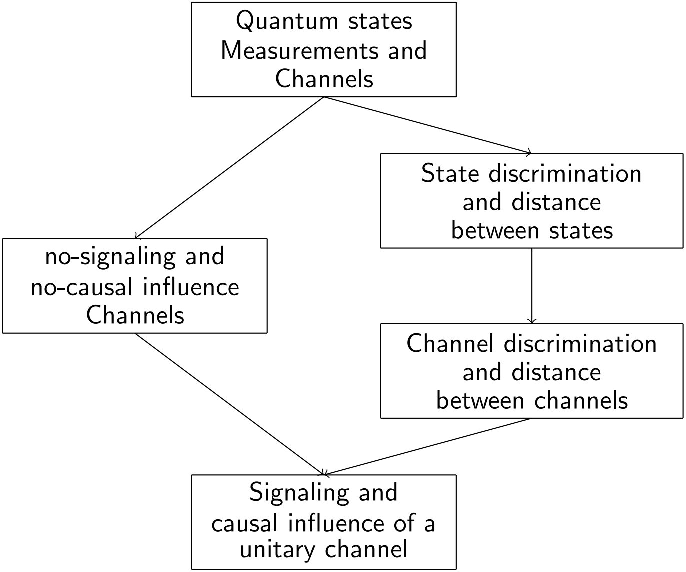
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Signaling and causal influence of a unitary channel

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any quantum state is an operator satisfying such properties

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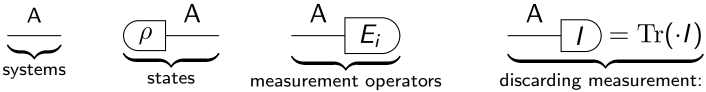
Reversible channels: $\mathcal{U} : \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{H})$ s.t. $\mathcal{U}\mathcal{U}^{-1} = \mathcal{U}^{-1}\mathcal{U} = \mathcal{I}$

$$\rho \to U\rho U^\dagger, \quad UU^\dagger = U^\dagger U = I$$

Diagrammatic notation

States and measurements



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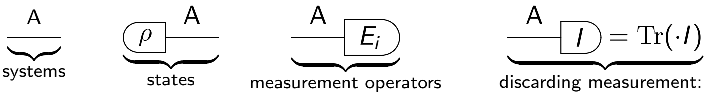

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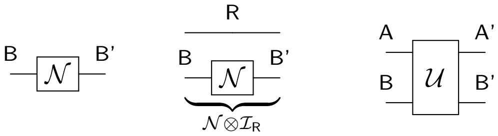

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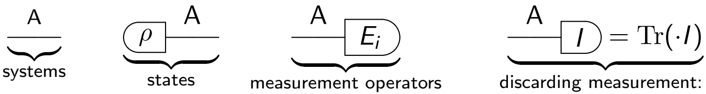

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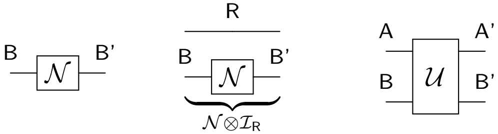
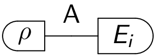
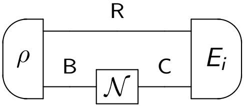


Combining states, measurements and channels

$P(i|\mathbf{E}, \rho) =:$ 

$P(i|\mathbf{E}, \mathcal{N}, \rho) =:$ 

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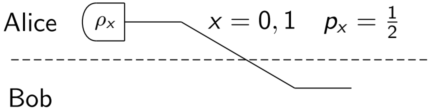
Scenario:



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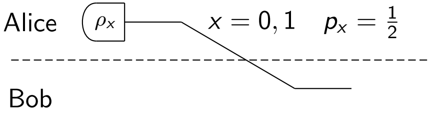

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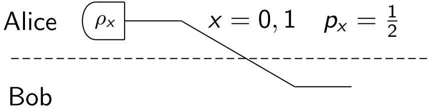

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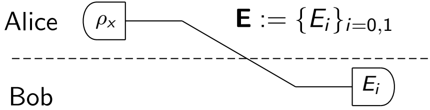

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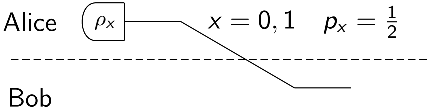

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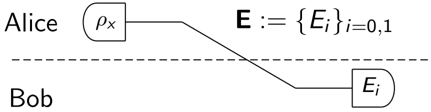

He declares that the state is

- ρ_0 if the outcome is 0;
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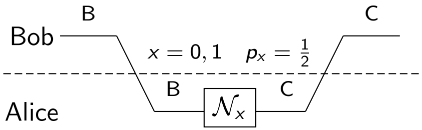

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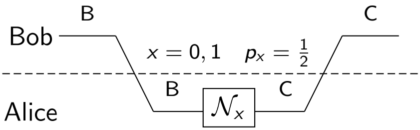

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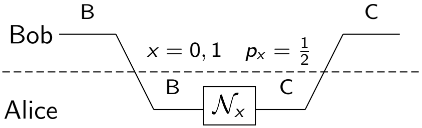

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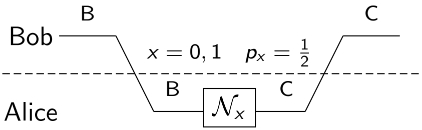

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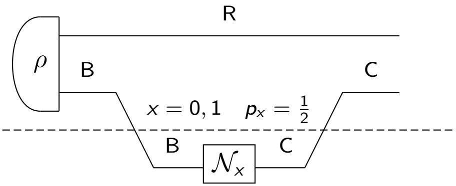

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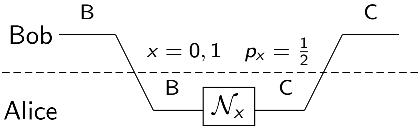

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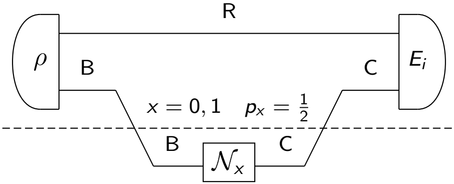

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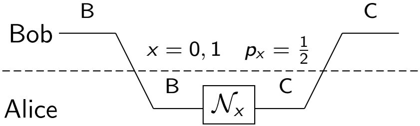

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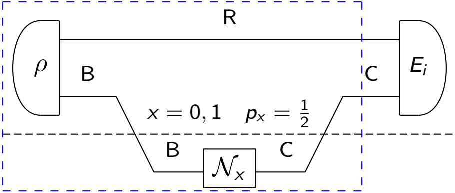

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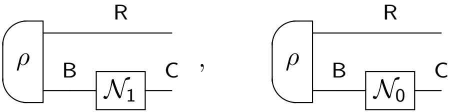

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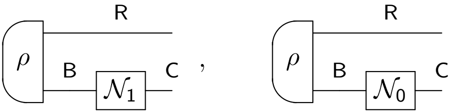

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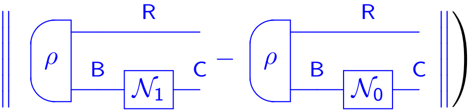

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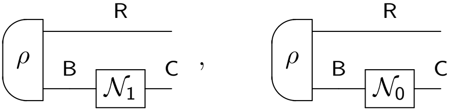

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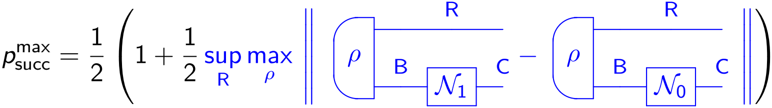

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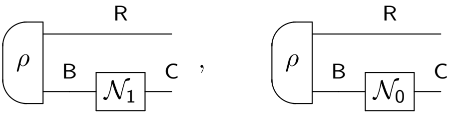

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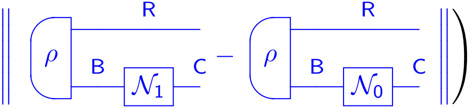
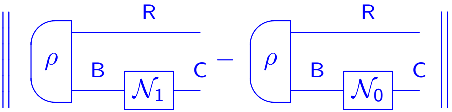
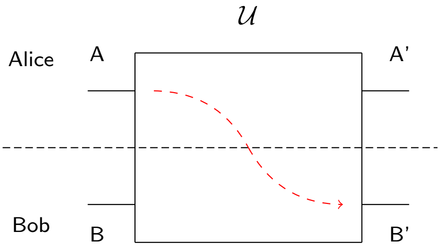


The diamond distance is defined as

$$\|\mathcal{N}_1 - \mathcal{N}_2\|_{\diamond} := \sup_{R}\max_{\rho}$$



Introduction and no-signaling channels



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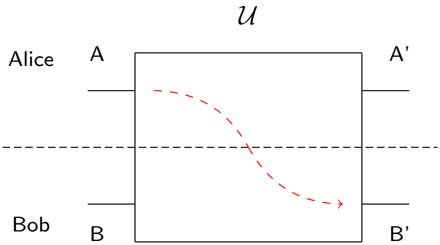

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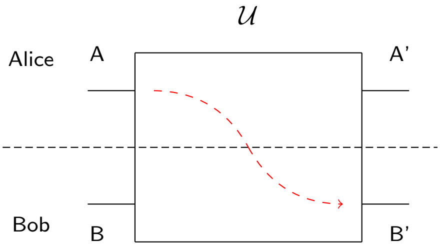

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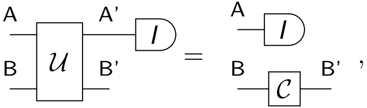
[Schumacher, Westmoreland QIP 4, 05]

no-signaling from A to B' if:

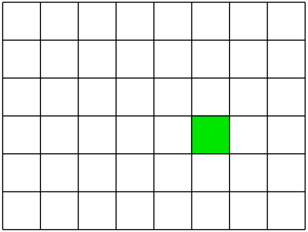


No-causal influence channels

For each site $\square = \mathcal{H}_\ell \leftrightarrow \mathcal{A}_\ell$

Cellular Automata:

$U : \otimes_\ell \mathcal{H}_\ell \to \otimes_\ell \mathcal{H}_\ell$.



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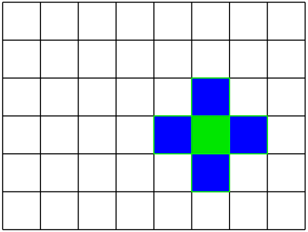

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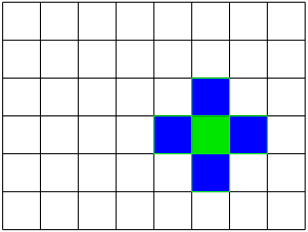
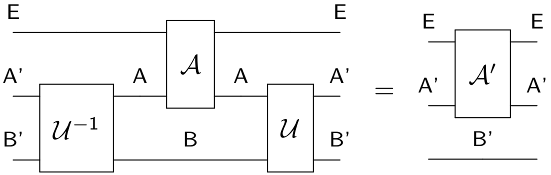


[Perinotti, Quantum 4, 20, Perinotti, Quantum 5, 21]

No-causal influence from A to B'

$\forall \mathcal{A} \in \mathfrak{C}(\mathrm{EA})$



No-causal influence gate

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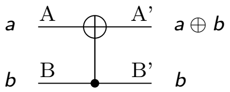

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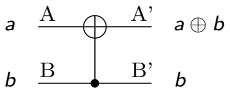

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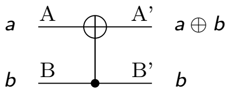

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It allows the creation of correlations between Alice and Bob at the output depending on the Alice's input.

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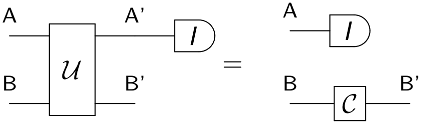
3rd question: is there a remnant of their inequivalence in QT?

Definition of signalling

[Barsse, Perinotti, Tosini, LV, PRR 6, 043305, 24]

Negate the no-signalling condition

$\exists \mathcal{C} \in \mathfrak{C}(\mathrm{B}, \mathrm{B}')$ s.t.



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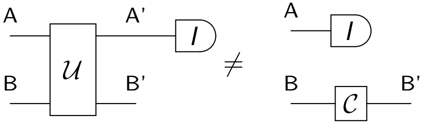

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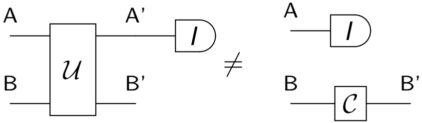

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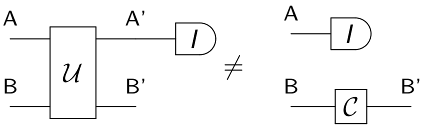

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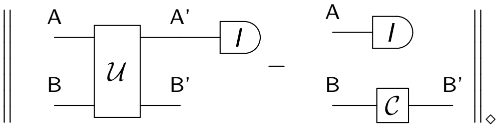

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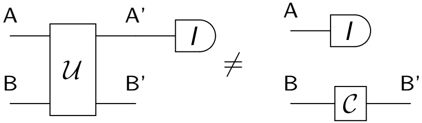

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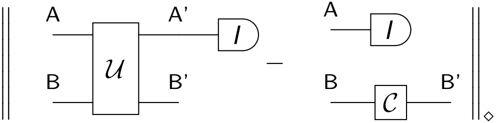
$\inf_{\mathcal{C}\in\mathfrak{C}(B,B')}$



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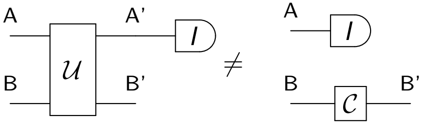

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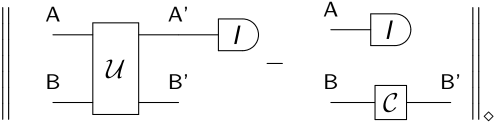
$$\Sigma(\mathcal{U}) := \inf_{\mathcal{C}\in\mathfrak{C}(B,B')}$$



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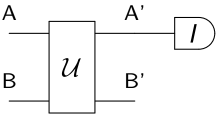
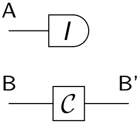


$$\neq$$



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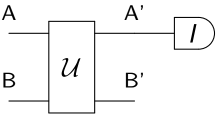

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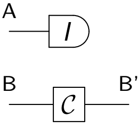


$$\Bigg\|_{\diamond}$$

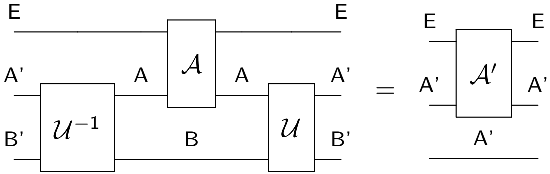
$\mathcal{U}$ is no-signalling $\iff$ $\Sigma(\mathcal{U}) = 0$;

$\Sigma(\mathcal{U}) \leq 2$

Equivalent condition for no causal influence

The actual definition of a no-causal influence channel is "useless"

$\forall \mathcal{A} \in \mathfrak{C}(\mathrm{EA})$



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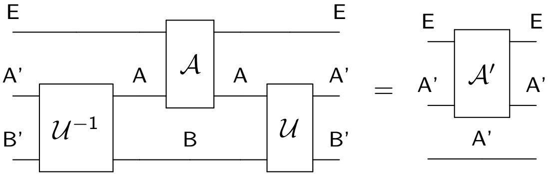
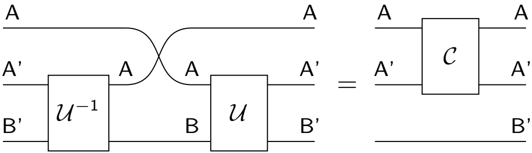


But it is sufficient to check the condition for the swap operator only:

$\exists \mathcal{C} \in \mathfrak{C}(\mathrm{EA}', \mathrm{EA}')$

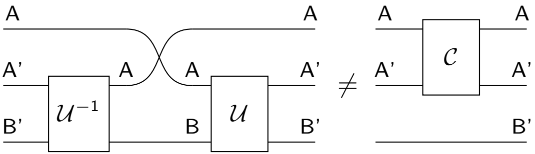


[Perinotti, Quantum 5, 21]

Causal influence of a unitary channel

Negate the alternative condition for no-causal influence

$\forall \mathcal{C} \in \mathfrak{C}(\mathrm{AA}')$ such that



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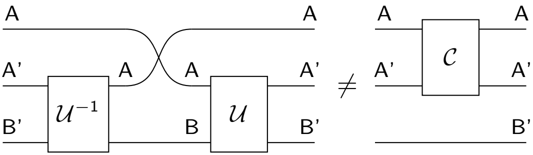

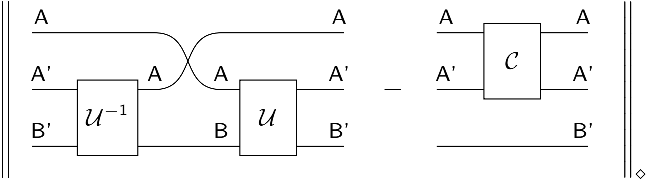

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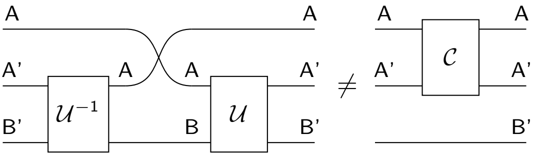

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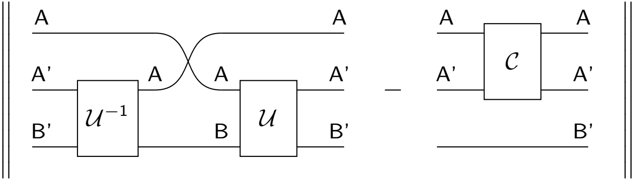

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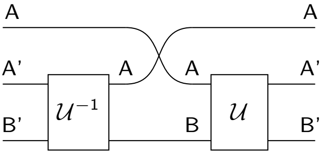

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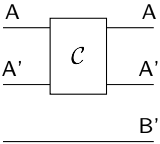
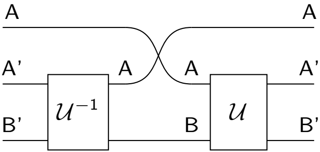


$C(\mathcal{U}) := \inf_{\mathcal{C} \in \mathfrak{C}(AA')}$ $\Bigg\|$



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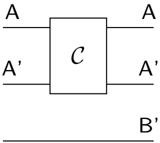

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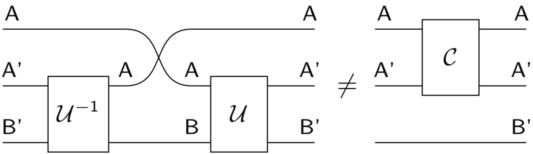

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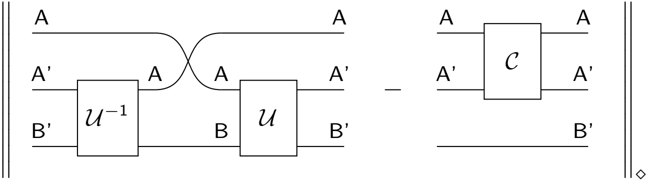

$\mathcal{U}$ does not induce causal influence from A to B' $\iff$ $C(\mathcal{U}) = 0$;

$C(\mathcal{U}) \leq 2$.

2nd question: Is there a continuity relation between Σ and C?

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Continuity relation between Σ and C

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1. if $C(\mathcal{U}) \leq \varepsilon$ then $\Sigma(\mathcal{U}) \leq ?$

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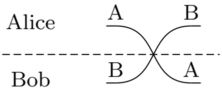
2. if $\Sigma(\mathcal{U}) \leq \varepsilon$ then $C(\mathcal{U}) \leq ?$

$$C(\mathcal{U}) \leq 2\sqrt{2}\Sigma(\mathcal{U})^{\frac{1}{2}} \leq 2\sqrt{2\varepsilon}$$

   small signaling $\implies$ small causal influence.

3rd question: Can we spot the inequivalence between signalling and causal influence even in quantum theory?

Examples: Swap and CNOT



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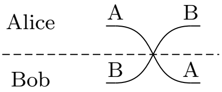

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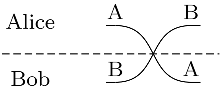

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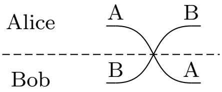

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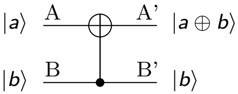

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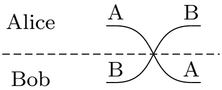

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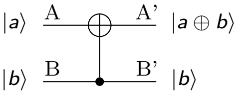

$C(\mathrm{CNOT}) = 2$, $\Sigma(\mathrm{CNOT}) = 1$

Recap

[Barsse, Perinotti, Tosini, LV, arXiv:2505.14120, 25]

	CNOT	Swap $d_{\mathrm{A}} = 2$	Swap	$\mathrm{CNOT}^{\otimes n}$
$\Sigma(\mathcal{U})$	1	$\frac{3}{2}$	$2\frac{d_{\mathrm{A}}^2-1}{d_{\mathrm{A}}^2}$	$2\frac{2^n-1}{2^n}$
$C(\mathcal{U})$	2	2	2	2

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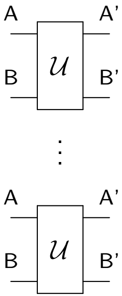
- The Swap achieves the maximum value for both signalling and causal influence;
- The CNOT achieves the maximum value for the causal influence, but not for the signalling.

Beyond the single-shot scenario

[Barsse, Perinotti, Tosini, LV, arXiv:2505.14120, 25]



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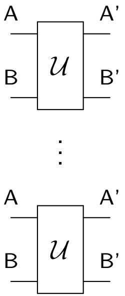

For all unitary channels $\mathcal{U}$ and $\mathcal{V}$,

$$\Sigma(\mathcal{U}\otimes\mathcal{V}) \geq \max\{\Sigma(\mathcal{U}),\Sigma(\mathcal{V})\}$$
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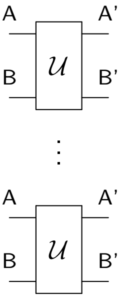

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$\Sigma(\mathcal{U}^{\otimes n})$ and $C(\mathcal{U}^{\otimes n})$ converge:

$$\Sigma_\infty(\mathcal{U}) := \lim_{n\to+\infty} \Sigma(\mathcal{U}^{\otimes n})$$

$$C_\infty(\mathcal{U}) := \lim_{n\to+\infty} C(\mathcal{U}^{\otimes n})$$

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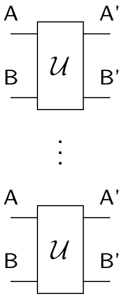

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$$C(\mathcal{U} \otimes \mathcal{V}) \geq \max\{C(\mathcal{U}), C(\mathcal{V})\}.$$

$\Sigma(\mathcal{U}^{\otimes n})$ and $C(\mathcal{U}^{\otimes n})$ converge:

$$\Sigma_\infty(\mathcal{U}) := \lim_{n \to +\infty} \Sigma(\mathcal{U}^{\otimes n})$$
$$C_\infty(\mathcal{U}) := \lim_{n \to +\infty} C(\mathcal{U}^{\otimes n})$$

$$\Sigma_\infty(\mathcal{S}) = \Sigma_\infty(\mathcal{C}_X) = 2$$

Conclusions

1. We have introduced two quantifiers to assess the amount of signalling and causal influence induced by a unitary channel;
2. we have shown that (in QT) if a unitary allows a small amount of signaling than it produces a small amount of causal influence (and viceversa);
3. We have computed such quantities in two explicit cases (Swap and CNOT);
4. We have studied the monotonicity of Σ and C under tensor product.