LIS - CANA seminar (Marseille)
12.11.2025

Antoine Soulas (aka Choups)
IQOQI Vienna
(Brukner's group)

On the emergence of preferred structures in quantum theory

Joint work with Guilherme Franzmann & Andrea Di Biagio
(soon on arXiv)

INTRODUCTION

- Initial motivations
  - Hilbert space fundamentalism (HSF) [Carroll & Singh, 2019]

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- Solving the tension has deep implications for **the notion of emergence in QM** **(in physics?)**

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SUMMARY

I. Cotler *et al.*'s theorem

1. K-locality
2. K-duality
3. The theorem

II. Stoica's theorem

1. Core idea
2. K-locality is unitary-invariant
3. H has more symmetries than the TPS
4. The time-evolution symmetry

III. How structures acquire their identity

1. Which notion of uniqueness to keep?
2. Insights from invariant theory
3. Can (H, |ψ>) uniquely determine a TPS?

I. COTLER *ET AL.*'S THEOREM

I.1. K-locality

- In all the talk, we fix integers n and $(d_i)_{1\leq i\leq n}$ with $d_i \geq 2$ and a Hilbert space $\mathcal{H}$ of finite dimension $\prod_i d_i$.

Definition (Tensor product structure). *A TPS of $\mathcal{H}$ is an equivalence class of isomorphisms $T : \mathcal{H} \to \bigotimes_{i=1}^{n} \mathcal{H}_i$ that factorize $\mathcal{H}$ into n factors $\mathcal{H}_i$ of respective dimensions d_i, where two isomorphisms T_1 and T_2 are said to be equivalent (denoted $T_1 \sim T_2$) if $T_1 T_2^{-1}$ is a product of local unitaries $U_1 \otimes \cdots \otimes U_n$ and arbitrary permutations of the factors.*

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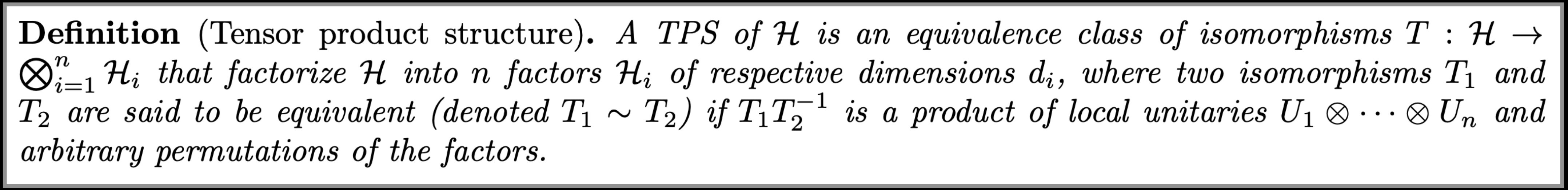

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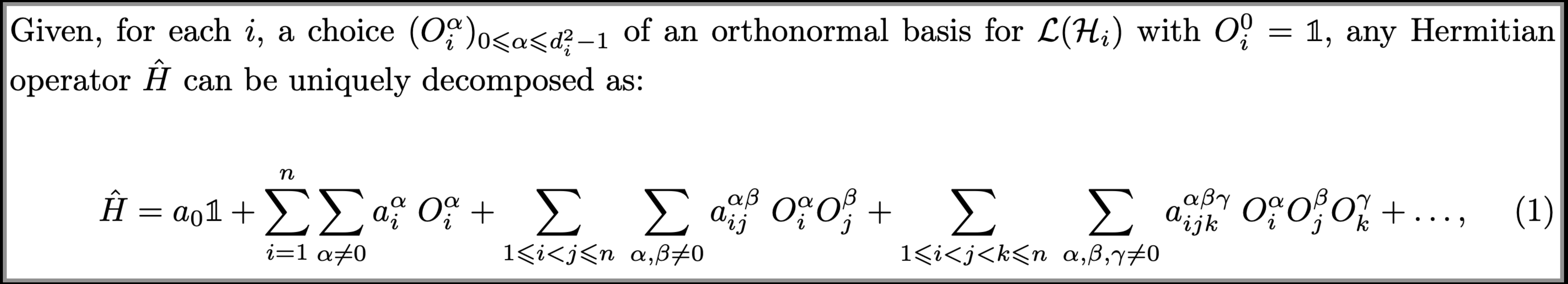
- H can be decomposed in a TPS:



Given, for each i, a choice $(O_i^\alpha)_{0\leqslant\alpha\leqslant d_i^2-1}$ of an orthonormal basis for $\mathcal{L}(\mathcal{H}_i)$ with $O_i^0 = \mathbb{1}$, any Hermitian operator $\hat{H}$ can be uniquely decomposed as:

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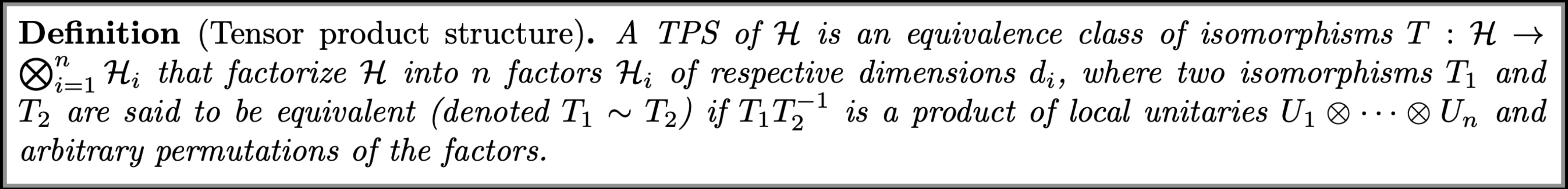

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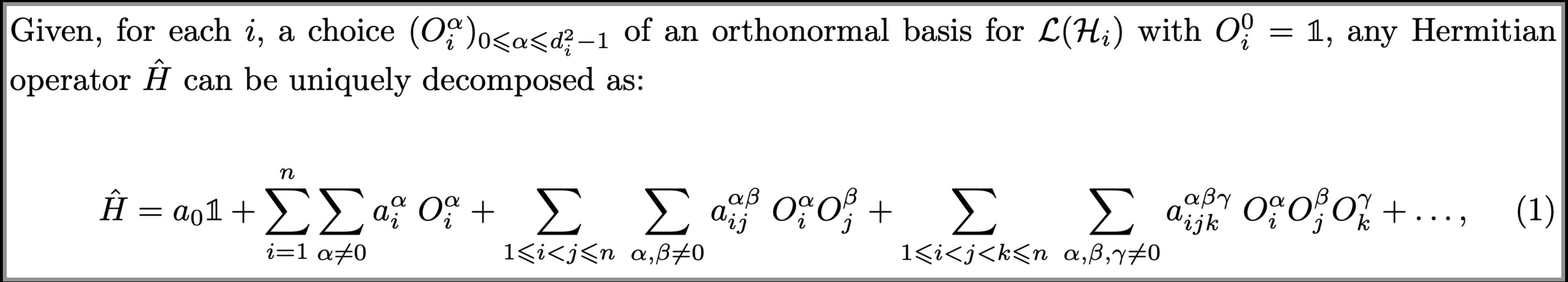

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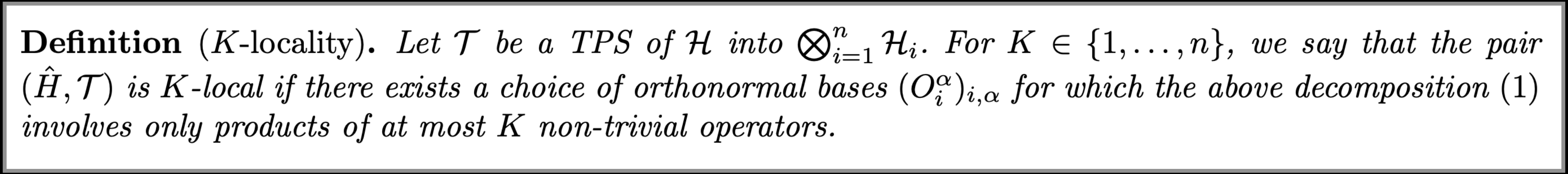
- Hence the following notion:



Definition (K-locality)**.** *Let $\mathcal{T}$ be a TPS of $\mathcal{H}$ into $\bigotimes_{i=1}^{n} \mathcal{H}_i$. For $K \in \{1, \ldots, n\}$, we say that the pair $(\hat{H}, \mathcal{T})$ is K-local if there exists a choice of orthonormal bases $(O_i^\alpha)_{i,\alpha}$ for which the above decomposition* (1) *involves only products of at most K non-trivial operators.*

Remark 1. K-locality vs. spacetime locality

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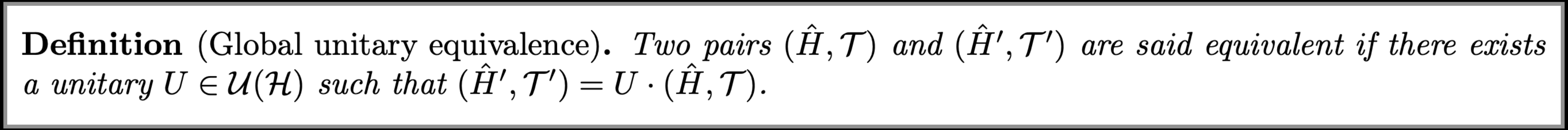
    - conversely, a QFT is always spacetime local, no matter K
- NB: most Hamiltonian are approx. 2-local in some TPS [Loizeau *et al.*, 2023]

N. Loizeau, F. Morone, and D. Sels. "Unveiling order from chaos by approximate 2-localization of random matrices", *Proceedings of the National Academy of Sciences* (2023)

I.2. K-duality

- For Cotler *et al.*, uniqueness of structures is understood **up to a global unitary**



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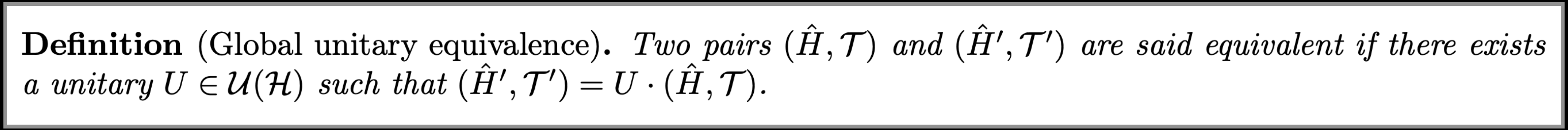

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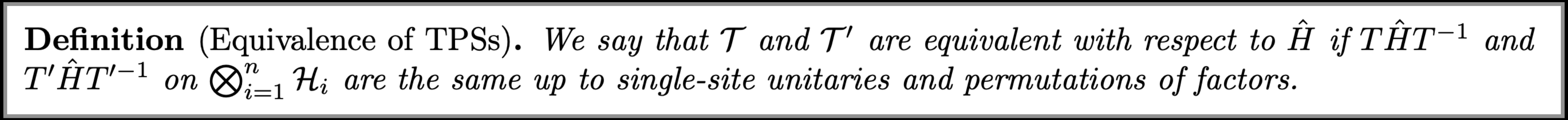

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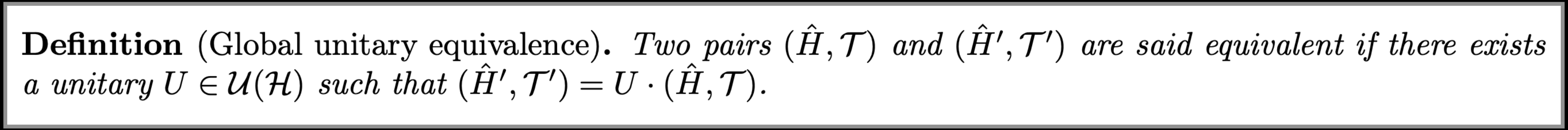

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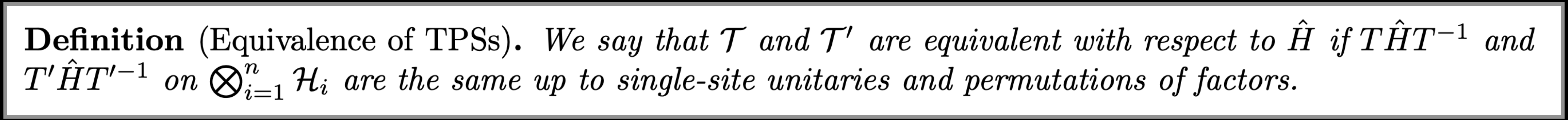

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- Whether H can uniquely determine a TPS can be phrased in terms of **dual TPSs**

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- **Example of dual TPSs**: Jordan-Wigner transform on the 1D Ising model

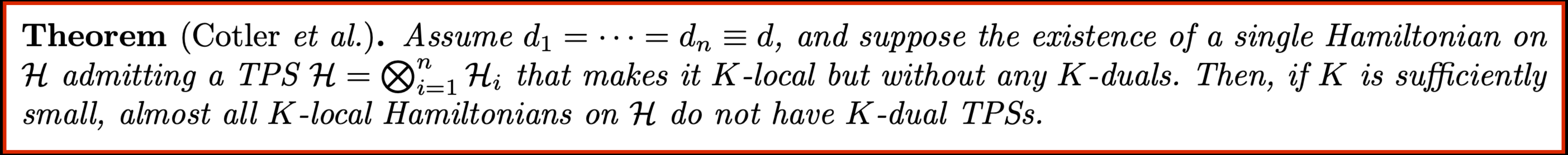
$$\mu_z^{(i)} = \prod_{j \leqslant i} \sigma_x^{(j)}$$
$$\mu_x^{(i)} = \sigma_z^{(i)} \sigma_z^{(i+1)}$$
$$\mu_x^{(n)} = \mu_z^{(n)}$$

→ induces

$$\hat{H} = \hat{H} = J \sum_i \sigma_z^{(i)} \sigma_z^{(i+1)} + h \sum_i \sigma_x^{(i)}$$
$$\downarrow$$
$$\hat{H} = J \sum_i \mu_x^{(i)} + h \sum_i \mu_z^{(i)} \mu_z^{(i+1)} - J\mu_x^{(n)} + h\mu_z^{(1)}$$

I.3. The theorem

- After a terribly intricate proof:



Theorem (Cotler *et al.*)**.** *Assume* $d_1 = \cdots = d_n \equiv d$, *and suppose the existence of a single Hamiltonian on* $\mathcal{H}$ *admitting a TPS* $\mathcal{H} = \bigotimes_{i=1}^{n} \mathcal{H}_i$ *that makes it* K*-local but without any* K*-duals. Then, if* K *is sufficiently small, almost all* K*-local Hamiltonians on* $\mathcal{H}$ *do not have* K*-dual TPSs.*

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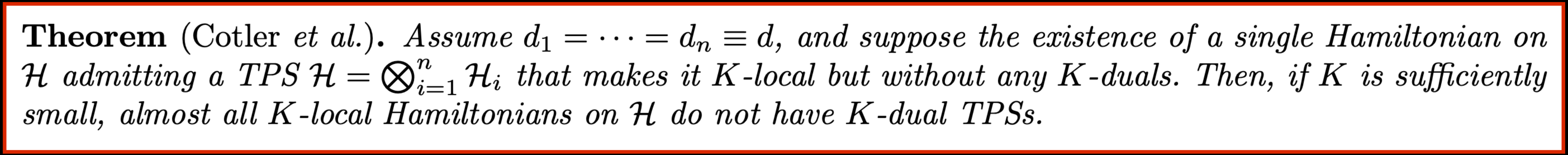

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- Numerical simulations to get convinced of the validity of the assumption.

II. STOICA'S THEOREM

II.1. Core idea

- Statement.

Theorem (Stoica)**.** *If a Hamiltonian* $\hat{H}$ *admits a TPS in which it is* K*-local, then such a TPS is not unique.*

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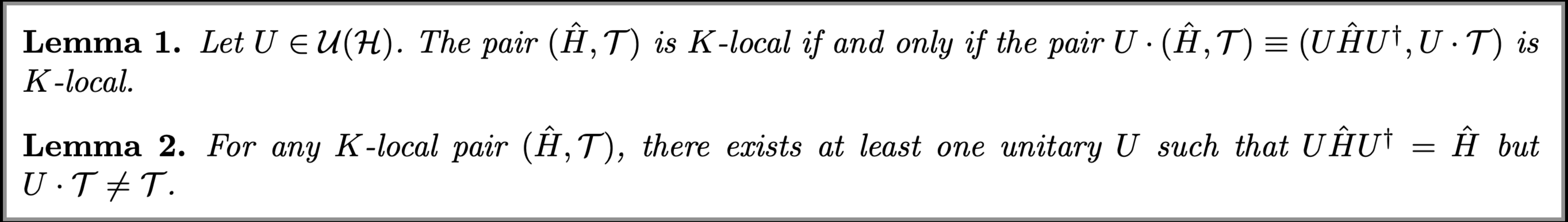
- Directly follows from



Lemma 1. *Let* $U \in \mathcal{U}(\mathcal{H})$*. The pair* $(\hat{H}, \mathcal{T})$ *is* K*-local if and only if the pair* $U \cdot (\hat{H}, \mathcal{T}) \equiv (U\hat{H}U^{\dagger}, U \cdot \mathcal{T})$ *is* K*-local.*

Lemma 2. *For any* K*-local pair* $(\hat{H}, \mathcal{T})$*, there exists at least one unitary* U *such that* $U\hat{H}U^{\dagger} = \hat{H}$ *but* $U \cdot \mathcal{T} \neq \mathcal{T}$*.*

II.2. K-locality is unitary-invariant

- Recall decomposition (1)

$$\hat{H} = a_0 \mathbb{1} + \sum_{i=1}^{n} \sum_{\alpha \neq 0} a_i^{\alpha}\, O_i^{\alpha} + \sum_{1 \leqslant i < j \leqslant n} \sum_{\alpha,\beta \neq 0} a_{ij}^{\alpha\beta}\, O_i^{\alpha} O_j^{\beta} + \sum_{1 \leqslant i < j < k \leqslant n} \sum_{\alpha,\beta,\gamma \neq 0} a_{ijk}^{\alpha\beta\gamma}\, O_i^{\alpha} O_j^{\beta} O_k^{\gamma} + \dots, \qquad (1)$$

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→ $UHU^{\dagger}$ decomposed in $U \cdot \mathcal{T}$ for the bases $(O_i^{\alpha})_{i,\alpha}$ has the same decomposition as H decomposed in $\mathcal{T}$ for the bases $(U\, O_i^{\alpha} U^{\dagger})_{i,\alpha}$

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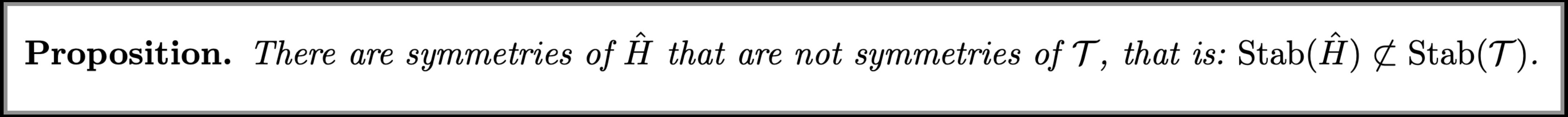

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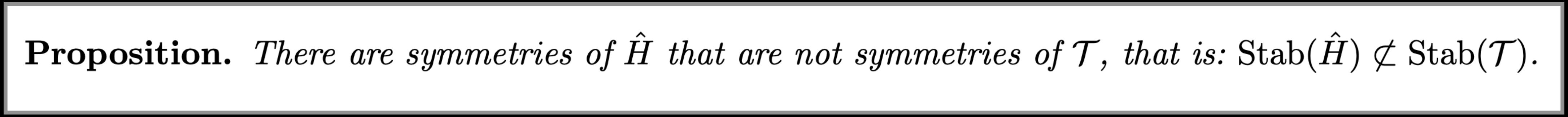

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- General proof (for any n): look at the commutative subgroups

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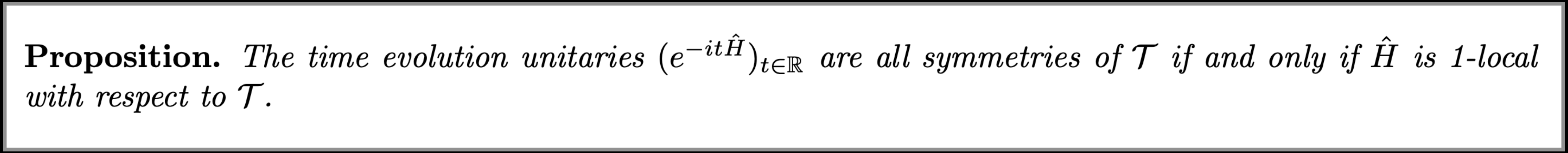

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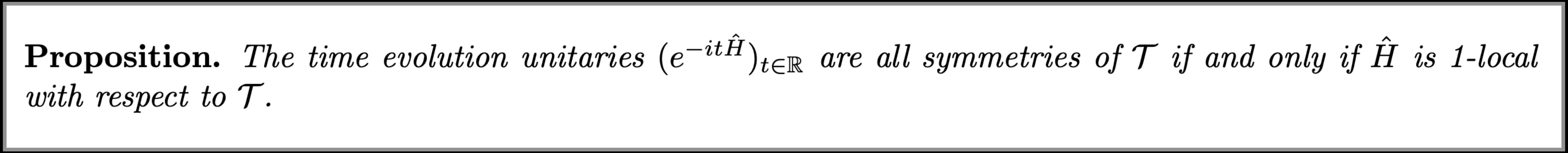

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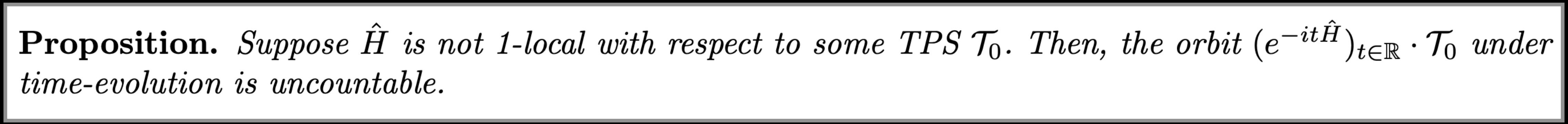

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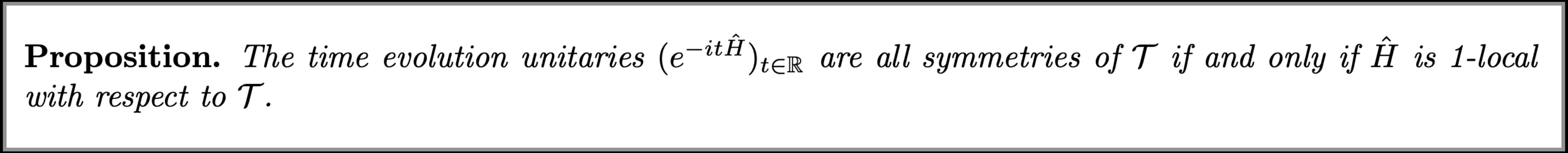

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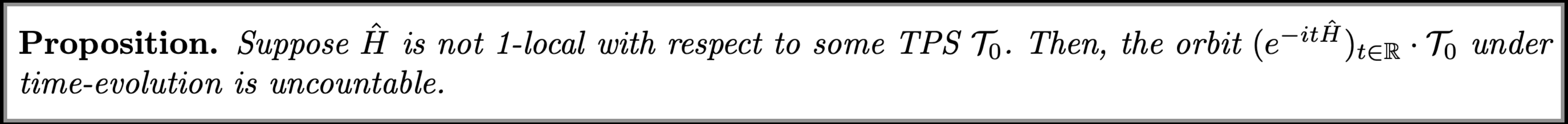

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    4. t → S[$\rho_i(t)$] takes an uncountable number of values, so $(e^{-iHt})_t \cdot \mathcal{T}_0$ sweeps an continuous orbit of TPSs

III. HOW STRUCTURES ACQUIRE THEIR IDENTITY

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→ **the relevant notion of uniqueness in physics is relational**

NB: also in maths!

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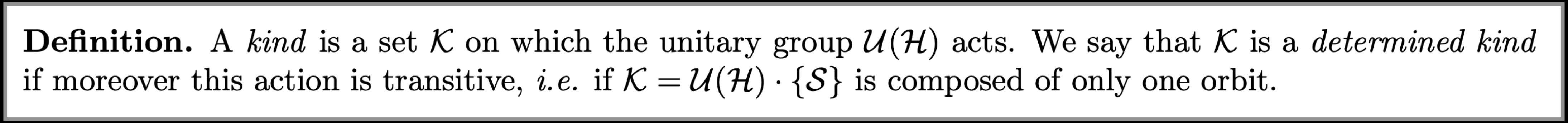
$$\boxed{P(\mathcal{S}_0, \mathcal{S}_e) \text{ and } P(\mathcal{S}'_0, \mathcal{S}'_e) \Rightarrow \exists U \in \mathcal{U}(\mathcal{H}) : (\mathcal{S}'_0, \mathcal{S}'_e) = U \cdot (\mathcal{S}_0, \mathcal{S}_e)}$$

III.2. Insights from invariants theory

- We need to talk about sets of structures of the same 'kind'.



Definition. A *kind* is a set $\mathcal{K}$ on which the unitary group $\mathcal{U}(\mathcal{H})$ acts. We say that $\mathcal{K}$ is a *determined kind* if moreover this action is transitive, *i.e.* if $\mathcal{K} = \mathcal{U}(\mathcal{H}) \cdot \{\mathcal{S}\}$ is composed of only one orbit.

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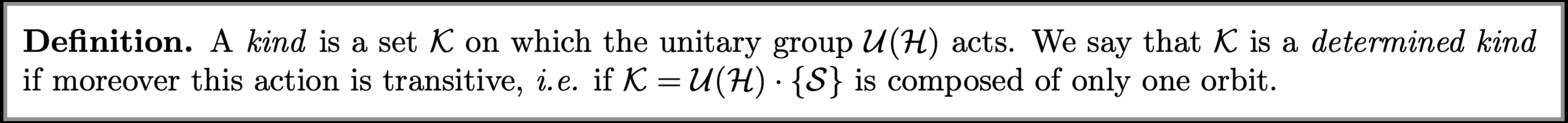

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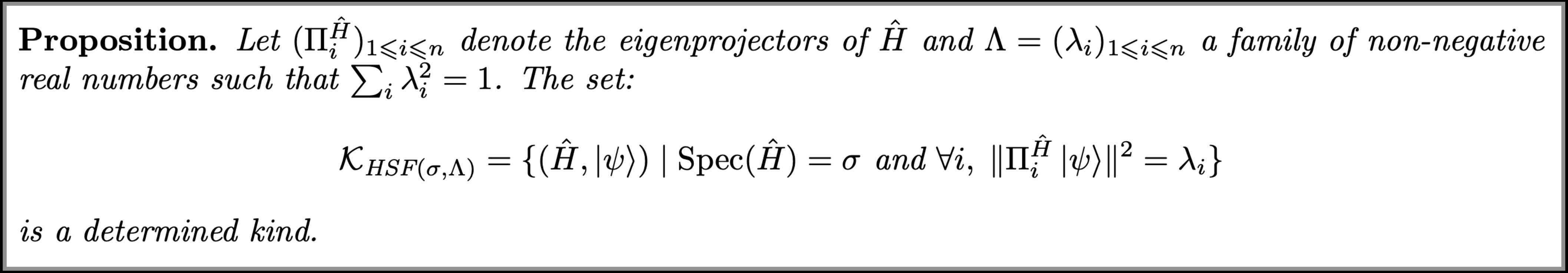

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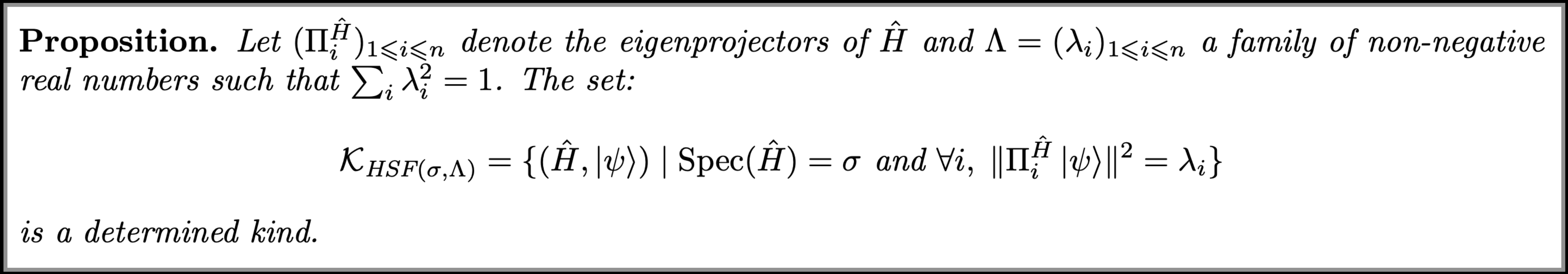

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- In fact, the unitary group acts **transitively and freely** on this kind.

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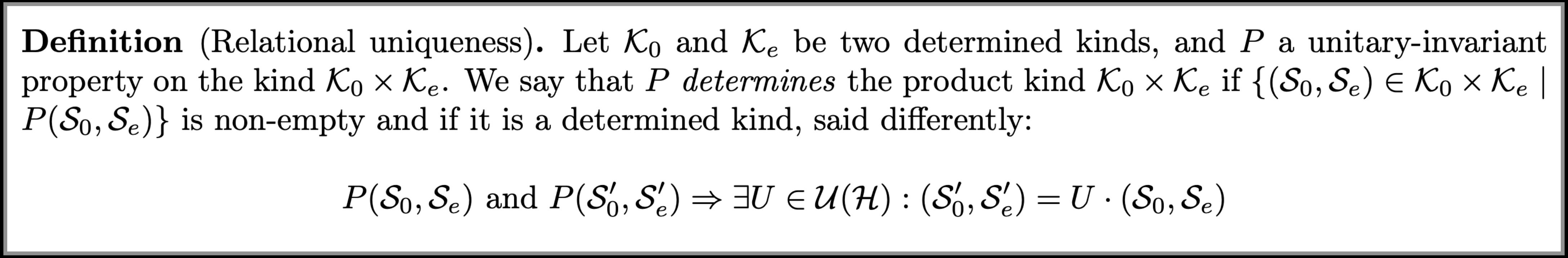

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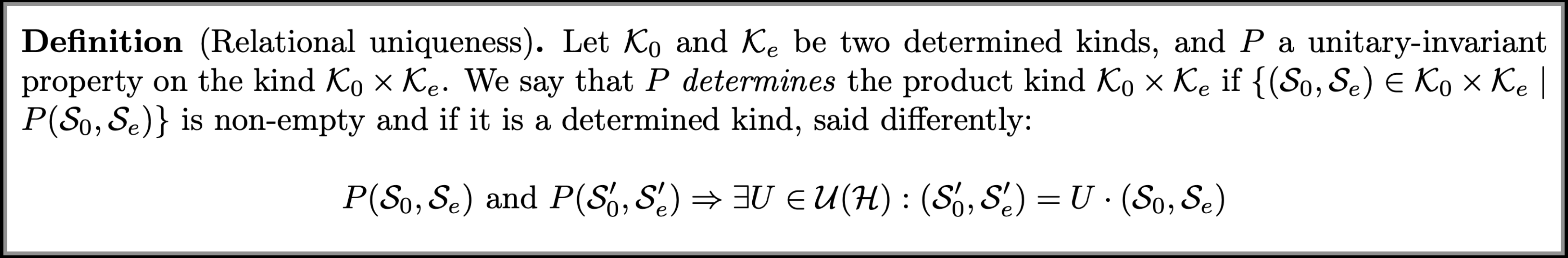

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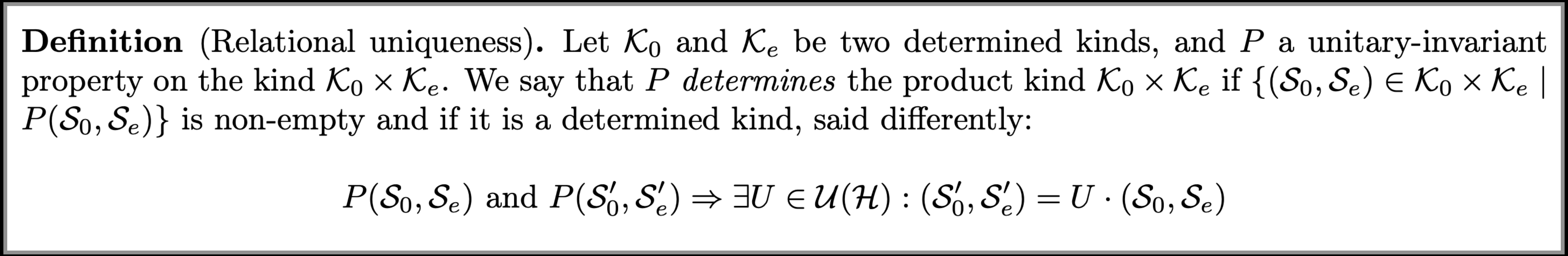

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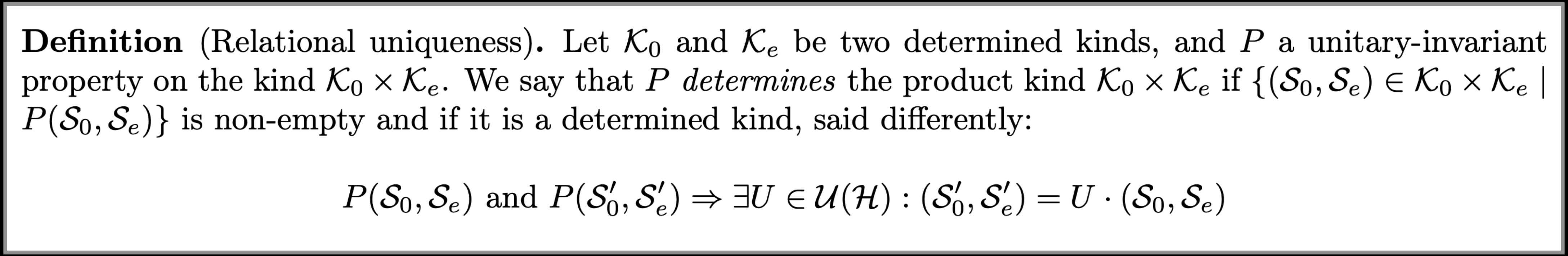

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  - recall the proof of II.4. (uncountable infinity of TPSs)

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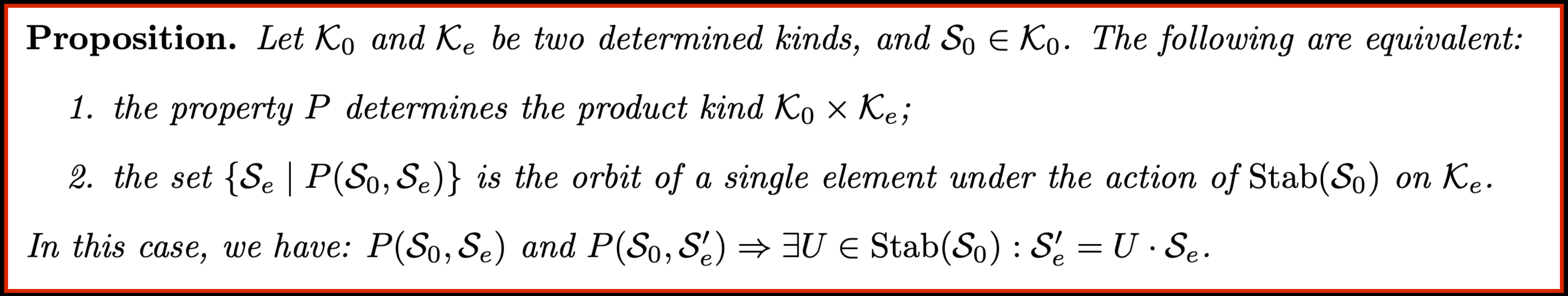
- Key result



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→ **applicable in the case of HSF** since Stab(H, |ψ>) = {1}

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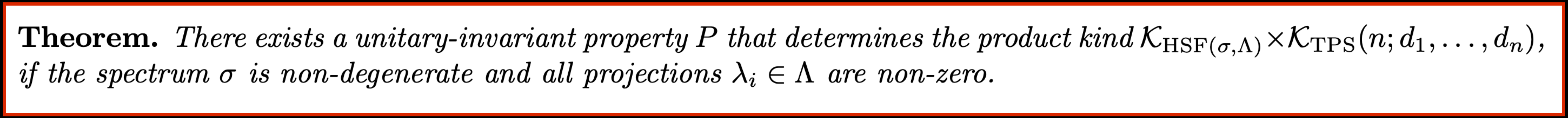

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    5. If P(H, lψ>, $\mathcal{T}$) and P(H, lψ>, $\mathcal{T}'$) hold, with $U \cdot \mathcal{T} = \mathcal{T}'$, then U maps product states of $\mathcal{T}$ to product states of $\mathcal{T}$

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  5. If P(H, lψ>, $\mathcal{T}$) and P(H, lψ>, $\mathcal{T}'$) hold, with U·$\mathcal{T}$=$\mathcal{T}'$, then U maps product states of $\mathcal{T}$ to product states of $\mathcal{T}$
  6. Conclude by the following lemma:

Lemma. *An operator* $A \in \mathcal{L}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n)$ *is a product of single-site operators* $A_1 \otimes \cdots \otimes A_n$ *if and only if* A *maps pure tensors to pure tensors.*

CONCLUSION

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- Selecting an emergent structure *via* a physical principle has proved successful.
  Examples: Einstein's tensor [Lovelock 1971], quantum fields, Schrödinger equation, bosonic and fermonic statistics [Mekonnen *et al.*, 2025]…)

F. Klein, "Vergleichende Betrachtungen über neuere geometrische Forschungen", *Mathematische Annalen* (1893)
D. Lovelock, "The Einstein tensor and its generalizations", *Journal of Mathematical Physics* (1971)
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Thank you for your attention!

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