Reaction Systems and Irreducible dynamics

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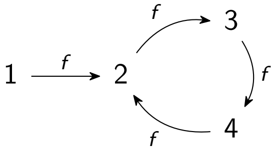
Finite Discrete Dynamical Systems

Definition

A *Finite Discrete Dynamical System* is a finite set D with a transition function $f : D \to D$.



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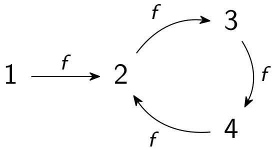

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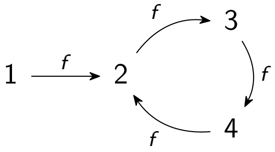
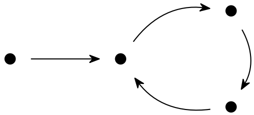
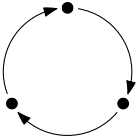


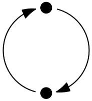
$\Rightarrow$ directed graphs with outdegree one

Up to isomorphism:

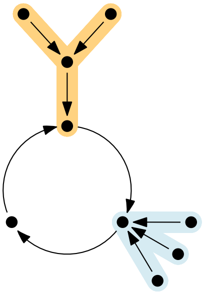
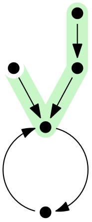


Limit cycles



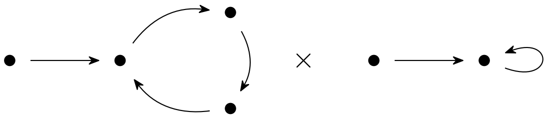


Limit cycles with trees going in.

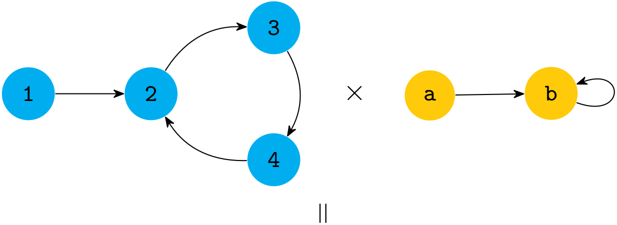



Product of Dynamical Systems

The product represents the *synchronous* evolution of the two systems.



Product of Dynamical Systems



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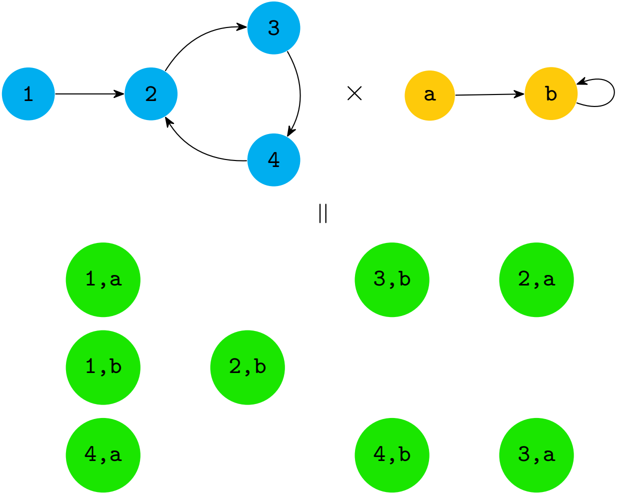

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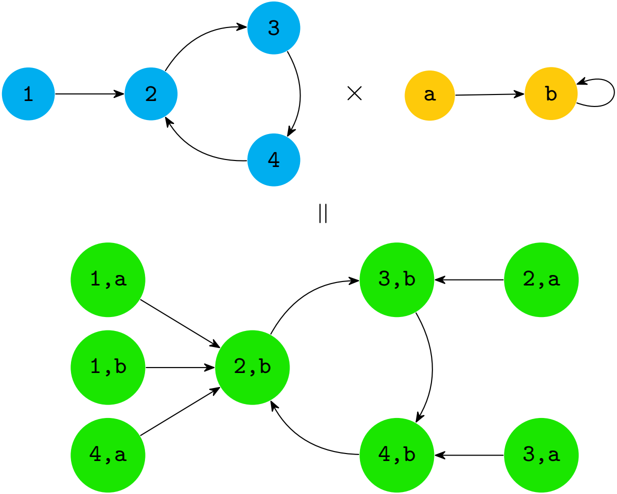
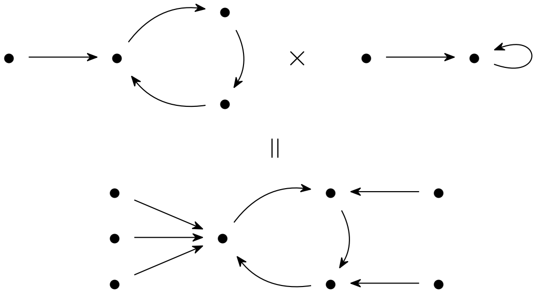


Product of Dynamical Systems

Up to isomorphism



Irreducibility

Remark that the dynamical system $\bullet \circlearrowleft$ is the identity element for the product, i.e.

$$\bullet \circlearrowleft \times \mathcal{D} = \mathcal{D}$$

Definition (Ir-reducible)

A dynamical system $\mathcal{D}$ is *reducible* if there exist $\mathcal{D}_1 \neq \bullet \circlearrowleft$ and $\mathcal{D}_2 \neq \bullet \circlearrowleft$ such that $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2$.
If $\mathcal{D}$ is not reducible, then it is called *irreducible*.

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Not unique factorization

$$(\bullet \circlearrowleft \quad \bullet \circlearrowleft) \times \bullet \rightleftarrows \bullet = \bullet \rightleftarrows \bullet \times \bullet \rightleftarrows \bullet$$

Irreducibility

Irreducible Dynamics

Input: A finite discrete dynamical system $\mathcal{D}$.

Question: Is $\mathcal{D}$ irreducible?

Irreducibility

Irreducible Dynamics
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Question: Is $\mathcal{D}$ irreducible?

We will study this problem in the context of Reaction Systems.

Reactions

Definition (Reaction)

Given a finite set S, a *reaction* a over S is a triple (R, I, P) of subsets of S:

- R is the set of *reactants*,
- I is the set of *inhibitors*,
- P is the set of *products*.

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- R is the set of *reactants*,
- I is the set of *inhibitors*,
- P is the set of *products*.

Definition (Enabled)

(R, I, P) is *enabled* in a *state* $T \subseteq S$ when:

$$R \subseteq T \quad \text{and} \quad I \cap T = \varnothing.$$

Reaction Systems

Definition (Reaction System)

A *reaction system* (RS) is a pair $\mathcal{A} = (S, A)$ where:

- S is a finite set of *symbols* or *entities*, called the *background set*;
- A is a set of reactions over S.

A *state* of $\mathcal{A}$ is a subset of S.

Reaction Systems

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A *state* of $\mathcal{A}$ is a subset of S.

Any reaction system induces a *finite discrete dynamical system* where the state set is 2^S (all the subsets of S).

Example of RS

Background set: $S = \{\bigstar, \diamondsuit\}$

Set of reactions: $r_1 = (\varnothing, \{\bigstar, \diamondsuit\}, \{\diamondsuit\})$

$r_2 = (\{\diamondsuit\}, \{\bigstar\}, \{\bigstar, \diamondsuit\})$

Example of RS

Background set: $S = \{\bigstar, \Diamond\}$

Set of reactions: $r_1 = (\varnothing, \{\bigstar, \Diamond\}, \{\Diamond\})$

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State: $T = \{\Diamond\}$

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Background set: $S = \{\bigstar, \diamondsuit\}$

Set of reactions: $r_1 = (\varnothing, \{\bigstar, \diamondsuit\}, \{\diamondsuit\}) \leftarrow$

$r_2 = (\{\diamondsuit\}, \{\bigstar\}, \{\bigstar, \diamondsuit\})$

State: $T = \{\diamondsuit\}$

$$\varnothing \subseteq T, \quad \{\bigstar, \diamondsuit\} \cap T = \{\diamondsuit\} \quad \Rightarrow \quad \mathrm{res}_{r_1}(T) = \varnothing$$

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$$\{\diamondsuit\} \subseteq T, \quad \{\bigstar\} \cap T = \varnothing \quad \Rightarrow \quad \mathrm{res}_{r_2}(T) = \{\bigstar, \diamondsuit\}$$

Result function on T:

$$\mathrm{res}_{\mathcal{A}}(T) = \mathrm{res}_{r_1}(T) \cup \mathrm{res}_{r_2}(T)$$

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Result function on T:

$$\mathrm{res}_{\mathcal{A}}(T) = \mathrm{res}_{r_1}(T) \cup \mathrm{res}_{r_2}(T) = \varnothing \cup \{\bigstar, \diamondsuit\} = \{\bigstar, \diamondsuit\}$$

Example of RS

Background set: $S = \{\bigstar, \Diamond\}$

Set of reactions: $r_1 = (\varnothing, \{\bigstar, \Diamond\}, \{\Diamond\})$

$r_2 = (\{\Diamond\}, \{\bigstar\}, \{\bigstar, \Diamond\})$

State: $T = \{\Diamond\}$

$$\varnothing \subseteq T, \quad \{\bigstar, \Diamond\} \cap T = \{\Diamond\} \quad \Rightarrow \quad \mathrm{res}_{r_1}(T) = \varnothing$$

$$\{\Diamond\} \subseteq T, \quad \{\bigstar\} \cap T = \varnothing \quad \Rightarrow \quad \mathrm{res}_{r_2}(T) = \{\bigstar, \Diamond\}$$

Result function on T:

$$\mathrm{res}_{\mathcal{A}}(\{\Diamond\}) = \{\bigstar, \Diamond\}$$

Example of RS

Background set: $S = \{\bigstar, \diamondsuit\}$

Set of reactions: $r_1 = (\varnothing, \{\bigstar, \diamondsuit\}, \{\diamondsuit\})$

$r_2 = (\{\diamondsuit\}, \{\bigstar\}, \{\bigstar, \diamondsuit\})$

State: $T = \{\diamondsuit\}$

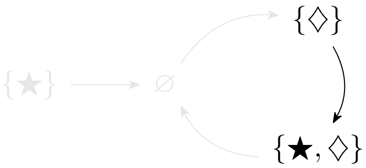
$$\varnothing \subseteq T, \quad \{\bigstar, \diamondsuit\} \cap T = \{\diamondsuit\} \quad \Rightarrow \quad \mathrm{res}_{r_1}(T) = \varnothing$$

$$\{\diamondsuit\} \subseteq T, \quad \{\bigstar\} \cap T = \varnothing \quad \Rightarrow \quad \mathrm{res}_{r_2}(T) = \{\bigstar, \diamondsuit\}$$

Result function on T:

$$\mathrm{res}_{\mathcal{A}}(\{\diamondsuit\}) = \{\bigstar, \diamondsuit\}$$

Representation of the dynamics:



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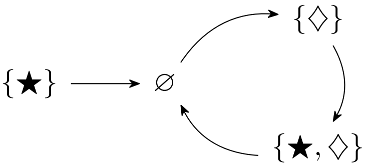
Dynamics of $\mathcal{A}$:

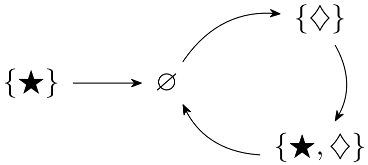


Example of RS
Dynamics of RS

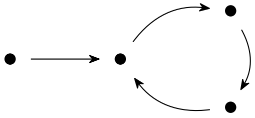
Definition

Given a RS $\mathcal{A} = (S, A)$, we will denote by $\mathcal{D}(\mathcal{A})$ its dynamics up to isomorphism.

Dynamics of $\mathcal{A}$:

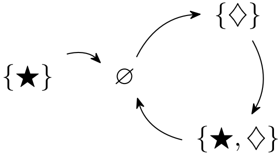


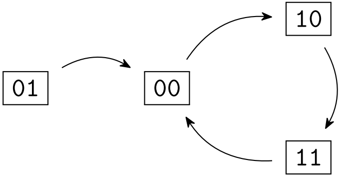
Dynamical system $\mathcal{D}(\mathcal{A})$ (up to isomorphism):



RS vs Boolean Automata Networks

$$x_\bigstar = \begin{cases} 1 \text{ if } \bigstar \in T \\ 0 \text{ if } \bigstar \notin T \end{cases} \qquad x_\diamondsuit = \begin{cases} 1 \text{ if } \diamondsuit \in T \\ 0 \text{ if } \diamondsuit \notin T \end{cases}$$

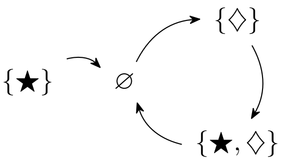


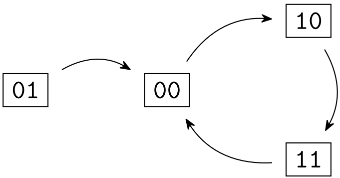


$(\varnothing, \{\bigstar, \diamondsuit\}, \{\diamondsuit\})$
$(\{\diamondsuit\}, \{\bigstar\}, \{\bigstar, \diamondsuit\})$

RS vs Boolean Automata Networks in DNF

$$x_{\bigstar} = \begin{cases} 1 \text{ if } \bigstar \in T \\ 0 \text{ if } \bigstar \notin T \end{cases} \qquad x_{\diamondsuit} = \begin{cases} 1 \text{ if } \diamondsuit \in T \\ 0 \text{ if } \diamondsuit \notin T \end{cases}$$





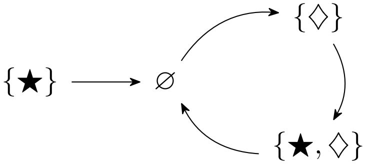
$(\varnothing, \{\bigstar, \diamondsuit\}, \{\diamondsuit\})$
$(\{\diamondsuit\}, \{\bigstar\}, \{\bigstar, \diamondsuit\})$

$f_{\diamondsuit}(x) = (\neg x_{\diamondsuit} \land \neg x_{\bigstar}) \lor (x_{\diamondsuit} \land \neg x_{\bigstar})$
$f_{\bigstar}(x) = x_{\diamondsuit} \land \neg x_{\bigstar}$

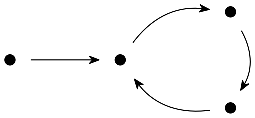
Example of RS
Cycle

Background set: $S = \{\bigstar, \Diamond\}$
Set of reactions: $r_1 = (\varnothing, \{\bigstar, \Diamond\}, \{\Diamond\})$
$r_2 = (\{\Diamond\}, \{\bigstar\}, \{\bigstar, \Diamond\})$

Dynamics of $\mathcal{A}$:



Dynamical system $\mathcal{D}(\mathcal{A})$:



Example of RS

Cycle

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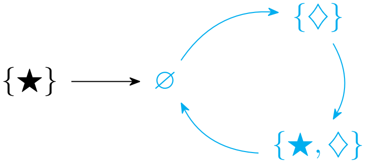
Dynamics of $\mathcal{A}$:



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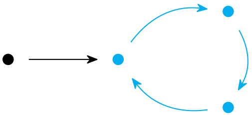
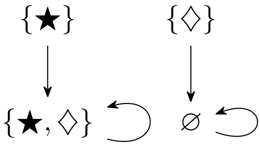


Properties of $\mathcal{D}(\mathcal{A})$: cycle global attractor (i.e. connected).

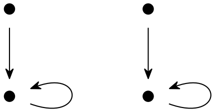
Example of RS
Fixed points

Background set: $S = \{\bigstar, \diamondsuit\}$
Set of reactions: $r_3 = (\{\bigstar\}, \varnothing, \{\bigstar \diamondsuit\})$

Dynamics of $\mathcal{B}$:



Dynamical system $\mathcal{D}(\mathcal{B})$:

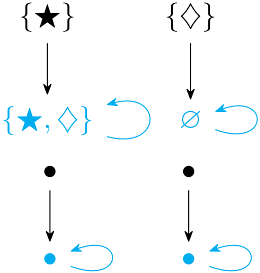


Example of RS
Fixed points

Background set: $S = \{\bigstar, \diamondsuit\}$
Set of reactions: $r_3 = (\{\bigstar\}, \varnothing, \{\bigstar\ \diamondsuit\})$

Dynamics of $\mathcal{B}$:

Dynamical system $\mathcal{D}(\mathcal{B})$:



Properties of $\mathcal{D}(\mathcal{B})$: two fixed points (therefore not connected).

Problems regarding dynamical properties

Given a reaction system $\mathcal{A} = (S, A)$:

- $\exists$ a fixed point? **NP**-complete[1]
- $\exists$ global cycle attractor? **PSPACE**-complete[2]
- ecc. . .

[1] Formenti, Manzoni, and Porreca 2014b.

[2] Formenti, Manzoni, and Porreca 2014a.

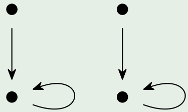
Irreducible Reaction Systems

Work in progress...

IRREDUCIBLE-RS
Input: A RS $\mathcal{A} = (S, A)$.
Question: Is $\mathcal{D}(\mathcal{A})$ irreducible?

Irreducible Reaction Systems

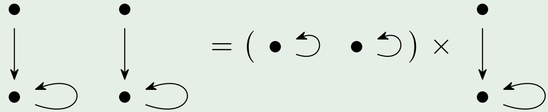
$\mathcal{B} = (\{\bigstar, \diamondsuit\}, \{r_3\})$, where $r_3 = (\{\bigstar\}, \varnothing, \{\bigstar, \diamondsuit\})$.
Is $\mathcal{D}(\mathcal{B})$ irreducible?

$\mathcal{D}(\mathcal{B}) =$ 

Irreducible Reaction Systems

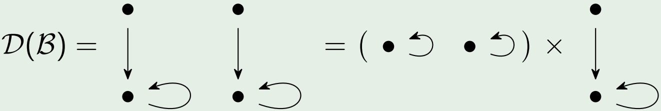
$\mathcal{B} = (\{\bigstar, \diamondsuit\}, \{r_3\})$, where $r_3 = (\{\bigstar\}, \varnothing, \{\bigstar, \diamondsuit\})$.

Is $\mathcal{D}(\mathcal{B})$ irreducible? No

$\mathcal{D}(\mathcal{B}) =$ 

Irreducible Reaction Systems

$\mathcal{B} = (\{\bigstar, \diamondsuit\}, \{r_3\})$, where $r_3 = (\{\bigstar\}, \varnothing, \{\bigstar, \diamondsuit\})$.
Is $\mathcal{D}(\mathcal{B})$ irreducible?



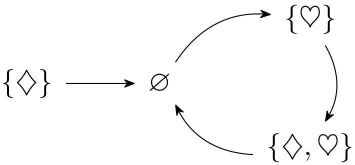
Indeed $\bullet \circlearrowleft \quad \bullet \circlearrowleft$ works exactly as $2 \in \mathbb{N}$.

Union of RSs

$\mathcal{A} = (S_{\mathcal{A}}, \{r_1, r_2\})$

$S_{\mathcal{A}} = \{\diamondsuit, \heartsuit\}$

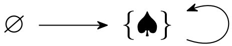
$r_1 = (\varnothing, \{\diamondsuit, \heartsuit\}, \{\heartsuit\})$
$r_2 = (\{\heartsuit\}, \{\diamondsuit\}, \{\diamondsuit, \heartsuit\})$



$\mathcal{B} = (S_{\mathcal{B}}, \{r_3\})$

$S_{\mathcal{B}} = \{\spadesuit\}$

$r_3 = (\varnothing, \varnothing, \{\spadesuit\})$

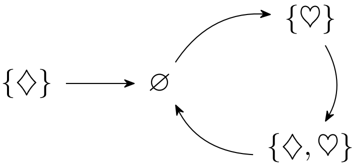


Union of RSs

$\mathcal{A} = (S_{\mathcal{A}}, \{r_1, r_2\})$

$S_{\mathcal{A}} = \{\diamondsuit, \heartsuit\}$

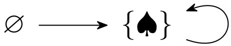
$r_1 = (\emptyset, \{\diamondsuit, \heartsuit\}, \{\heartsuit\})$
$r_2 = (\{\heartsuit\}, \{\diamondsuit\}, \{\diamondsuit, \heartsuit\})$



$\mathcal{B} = (S_{\mathcal{B}}, \{r_3\})$

$S_{\mathcal{B}} = \{\spadesuit\}$

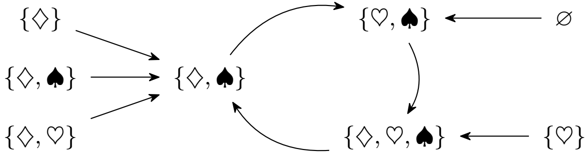
$r_3 = (\emptyset, \emptyset, \{\spadesuit\})$



What happens if we take the union of the two RSs?

Union of RSs. . .

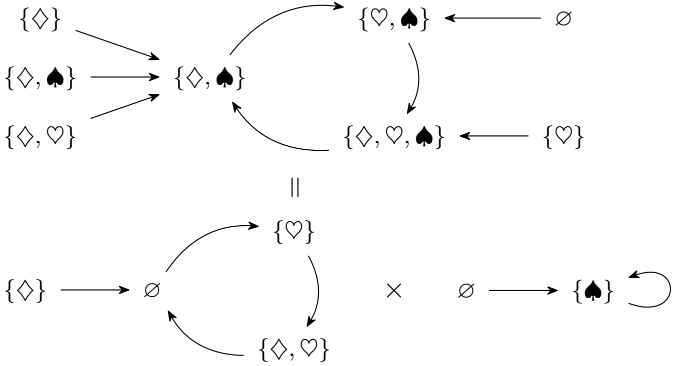
$$\mathcal{A} \sqcup \mathcal{B} = (\{\diamondsuit, \heartsuit, \spadesuit\}, \{r_1, r_2, r_3\})$$
$$r_1 = (\varnothing, \{\diamondsuit, \heartsuit\}, \{\heartsuit\})$$
$$r_2 = (\{\heartsuit\}, \{\diamondsuit\}, \{\diamondsuit, \heartsuit\})$$
$$r_3 = (\varnothing, \varnothing, \{\spadesuit\})$$



Union of RSs...

...gives the product of the dynamics!

$$\mathcal{A} \sqcup \mathcal{B} = (\{\diamondsuit, \heartsuit, \spadesuit\}, \{r_1, r_2, r_3\})$$
$$r_1 = (\varnothing, \{\diamondsuit, \heartsuit\}, \{\heartsuit\})$$
$$r_2 = (\{\heartsuit\}, \{\diamondsuit\}, \{\diamondsuit, \heartsuit\})$$
$$r_3 = (\varnothing, \varnothing, \{\spadesuit\})$$



Union of RSs. . .

. . . gives the product of the dynamics!

$$r_1 = (\varnothing, \{\diamondsuit, \heartsuit\}, \{\heartsuit\})$$

$$r_2 = (\{\heartsuit\}, \{\diamondsuit\}, \{\diamondsuit, \heartsuit\})$$

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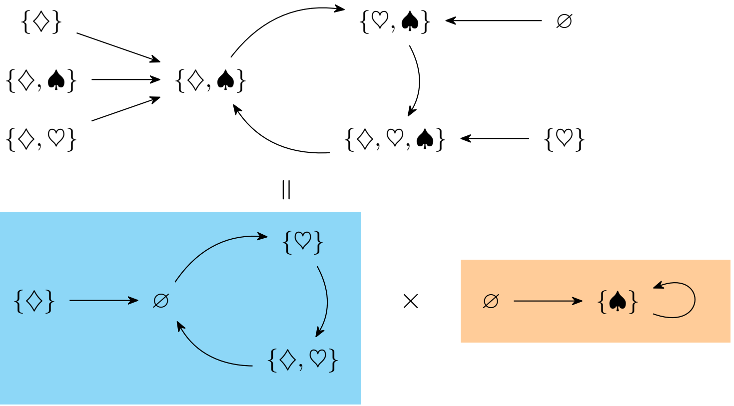


Rocco Ascone

FDDS

Reaction systems

Irreducibility in RS

Resource bounded RSs

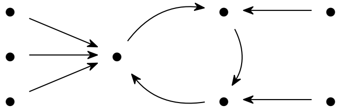
Bijective $\mathcal{RS}(\infty, 0)$

Future research

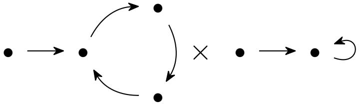
Union of RSs and product of their dynamics

Up to isomorphism:

$\mathcal{D}(\mathcal{A} \sqcup \mathcal{B}) =$



$\|$



$= \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{B})$

$$\mathcal{D}(\mathcal{A} \sqcup \mathcal{B}) = \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{B})$$

Lower bound

Theorem

IRREDUCIBLE-RS *is* **NP***-hard.*

Lower bound

Theorem

IRREDUCIBLE-RS *is* **NP***-hard.*

Open problem

Find an upper bound for IRREDUCIBLE-RS stronger than **NEXPTIME**.

Lower bound: Proof's Idea

NP-hardness

- φ CNF formula over $V = \{x_1, \ldots, x_n\}$.

Lower bound: Proof's Idea

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- Encode each assignment as a subset $X \subseteq V$.
  E.g. $X = \{x_1, x_3\}$ means $x_1 = 1$, $x_2 = 0$, $x_3 = 1$.
- Extra symbols $\{\diamondsuit, \spadesuit\}$.

Lower bound: Proof's Idea

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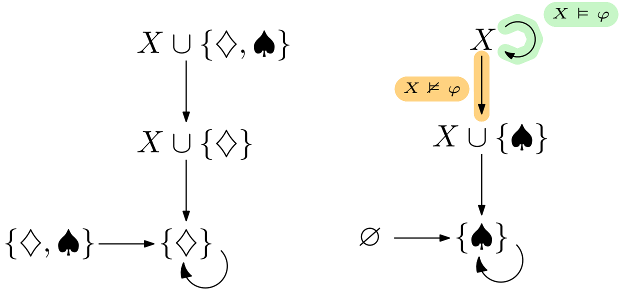
Local dynamics for each possible assignment $\varnothing \subsetneq X \subseteq V$:

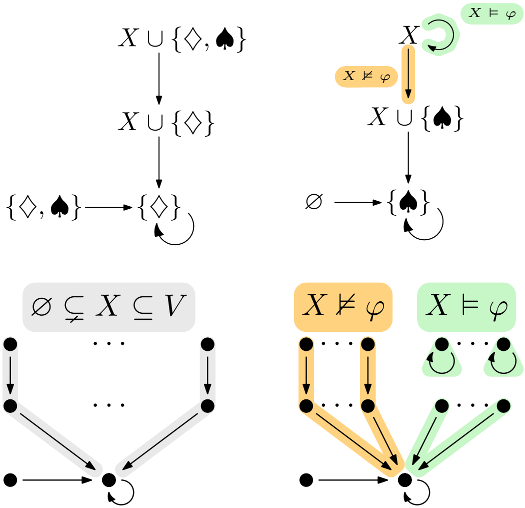


Lower bound: Proof's Idea

NP-hardness

Local dynamics:

Global dynamics:



Lower bound: Proof's Idea

NP-hardness

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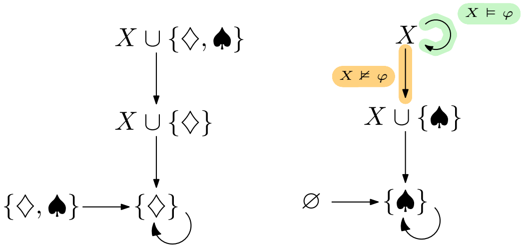
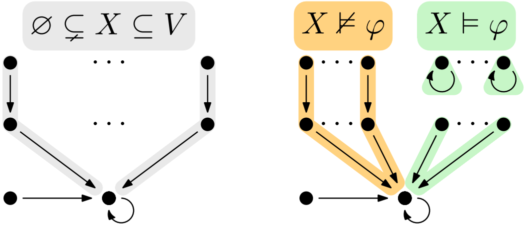


Global dynamics:



Example on: gitlab.lis-lab.fr/rocco.ascone/rs-irreducible

Divisibility in RSs is **NP**-hard

2-Divisibility-RS
Input: A RS $\mathcal{A} = (S, A)$.
Question: Is $\mathcal{D}(\mathcal{A})$ divisible by $2 = \bullet \circlearrowleft \quad \bullet \circlearrowleft$?

Corollary

Divisibility-RS *is* **NP**-*hard.*

Irreducibility in restricted classes of RSs

Research direction

Find non-trivial subclasses of RSs where the problem is easier to solve.

Irreducibility in restricted classes of RSs

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Find non-trivial subclasses of RSs where the problem is easier to solve.

Until now, we have found two non-trivial subclasses of RSs: bijective $\mathcal{RS}(\infty, 0)$ and bijective $\mathcal{RS}(0, \infty)$.

Resource-bounded systems

Class of RSs	Type of reactions
$\mathcal{RS}(\infty, \infty)$	$(\{a, b\}, \{c\}, \{\dots\})$
$\mathcal{RS}(0, \infty)$	$(\varnothing, \{a, b\}, \{\dots\})$
$\mathcal{RS}(\infty, 0)$	$(\{a, b\}, \varnothing, \{\dots\})$
$\mathcal{RS}(1, 0)$	$(\{a\}, \varnothing, \{\dots\})$

Resource-bounded systems: classification[3]

Class of RSs	Type of reactions	Subclass of $2^S \to 2^S$
$\mathcal{RS}(\infty, \infty)$	$(\{a, b\}, \{c\}, \{\dots\})$	all
$\mathcal{RS}(0, \infty)$	$(\varnothing, \{a, b\}, \{\dots\})$	antitone
$\mathcal{RS}(\infty, 0)$	$(\{a, b\}, \varnothing, \{\dots\})$	monotone
$\mathcal{RS}(1, 0)$	$(\{a\}, \varnothing, \{\dots\})$	additive

[3] Manzoni, Poças, and Porreca 2014.

Resource-bounded systems: classification

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Resource-bounded systems: classification

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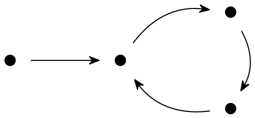
$\mathcal{RS}(\infty, 0)$	monotone: $T \subseteq T' \Rightarrow \mathrm{res}(T) \subseteq \mathrm{res}(T')$
$\mathcal{RS}(1, 0)$	additive: $\mathrm{res}(T \cup T') = \mathrm{res}(T) \cup \mathrm{res}(T')$

$\nexists$ global 3-cycle attractor in $\mathcal{RS}(0,\infty)$ or $\mathcal{RS}(\infty,0)$

The dynamical system:



can be obtained with a general RS.

On the other hand, it cannot be achieved with only reactions of the type $(\varnothing, \{\dots\}, \{\dots\})$ or only reactions of the type $(\{\dots\}, \varnothing, \{\dots\})$.

Results on Cycles and Global Attractors

Problem		$\mathcal{RS}(\infty,\infty)$	$\mathcal{RS}(0,\infty)$	$\mathcal{RS}(\infty,0)$
A given state is a global attractor		**PSPACE**-c[5]	**P**	**P**
$\exists$ global fixed point attractor		**PSPACE**-c[5]	**P**	**P**
$\exists$ global cycle attractor of length at least k	$k = 2$	**PSPACE**-c[6]	**PSPACE**-c	$\nexists$
	$k > 2$	**PSPACE**-c[6]	$\nexists$	$\nexists$
A given state is part of a cycle		**PSPACE**-c[5]	**PSPACE**-c	**PSPACE**-c
$\exists$ common cycle		**PSPACE**-c[5]	**PSPACE**-c	**PSPACE**-c
sharing all cycles		**PSPACE**-c[5]	**PSPACE**-c	**PSPACE**-c

5 Formenti, Manzoni, and Porreca 2014a

6 Dennunzio et al. 2019

Ascone, Bernardini, and Manzoni 2025

Results on Fixed Points

Problem	$\mathcal{RS}(\infty,\infty)$	$\mathcal{RS}(0,\infty)$	$\mathcal{RS}(\infty,0)$	$\mathcal{RS}(1,0)$
A given state is a fixed point attractor	**NP**-c[1]	**NP**-c[2]	**NP**-c[2]	**P** [4]
∃ fixed point	**NP**-c[1]	**NP**-c[2]	**P** [3]	
∃ common fixed point	**NP**-c[1]	**NP**-c[2]	**NP**-c[2]	**P** [4]
sharing all fixed points	**coNP**-c[1]	**coNP**-c[2]	**coNP**-c[2]	**P** [4]
∃ fixed point attractor	**NP**-c[1]	**NP**-c[2]	Unknown	**P** [4]
∃ common fixed point attractor	**NP**-c[1]	**NP**-c[2]	**NP**-c[2]	**P** [4]
sharing all fixed points attractor	$\mathbf{\Pi_2^P}$-c[1]	$\mathbf{\Pi_2^P}$-c[2]	$\mathbf{\Pi_2^P}$-c[2]	**P** [4]
∃ fixed point not attractor	$\mathbf{\Sigma_2^P}$-c[2]	$\mathbf{\Sigma_2^P}$-c[2]	$\mathbf{\Sigma_2^P}$-c[2]	**P** [4]
∃ common fixed point not attractor	$\mathbf{\Sigma_2^P}$-c[2]	$\mathbf{\Sigma_2^P}$-c[2]	$\mathbf{\Sigma_2^P}$-c[2]	**P** [4]
sharing all fixed points not attractor	**coNP**-c[2]	**coNP**-c[2]	**coNP**-c[2]	**P** [4]
$\mathrm{res}_\mathcal{A} = \mathrm{res}_\mathcal{B}$	**coNP**-c[2]	**P** [2]	**P** [2]	
res bijective	**coNP**-c [1]	**P** [2]	**P** [2]	

1 Formenti, Manzoni, and Porreca 2014b
2 Ascone, Bernardini, and Manzoni 2024b
3 Knaster-Tarski Theorem
4 Ascone, Bernardini, and Manzoni 2024a

Bijective $\mathcal{RS}(\infty, 0) \approx$ Bijective $\mathcal{RS}(1, 0)$

Proposition (Ascone, Bernardini, and Manzoni 2024b)

We can reduce any bijective $\mathcal{RS}(\infty, 0)$ to a bijective $\mathcal{RS}(1, 0)$ having the same dynamics.

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What does it mean *bijective* for $\mathcal{RS}(1, 0)$ in terms of their reactions?

- Each reaction is of the form $(\{x\}, \varnothing, \{y\})$ for $x, y \in S$.

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- The result function must be injective ($\Rightarrow$ bijective) on the singletons $\{x\}$, for $x \in S$.

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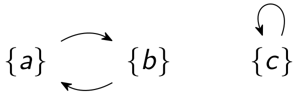
$(\{a\}, \varnothing, \{b\})$
$(\{b\}, \varnothing, \{a\})$ $\Rightarrow$
$(\{c\}, \varnothing, \{c\})$



is injective

Bijective $\mathcal{RS}(1,0)$ irreducibility

Is the following RS irreducible?

$(\{a\}, \varnothing, \{b\})$
$(\{b\}, \varnothing, \{a\})$
$(\{c\}, \varnothing, \{c\})$

Bijective $\mathcal{RS}(1,0)$ irreducibility

Is the following RS irreducible? No, by the argument above, we can split it into two smaller RSs $\Rightarrow$ the dynamics is reducible

$(\{a\}, \varnothing, \{b\})$

$(\{b\}, \varnothing, \{a\})$

$(\{c\}, \varnothing, \{c\})$

Bijective $\mathcal{RS}(1,0)$ irreducibility

Is the following RS irreducible?

$$\mathcal{D}\begin{pmatrix} (\{a\},\varnothing,\{b\}) \\ (\{b\},\varnothing,\{a\}) \\ (\{c\},\varnothing,\{c\}) \end{pmatrix} = \mathcal{D}\begin{pmatrix} (\{a\},\varnothing,\{b\}) \\ (\{b\},\varnothing,\{a\}) \end{pmatrix} \times \mathcal{D}\begin{pmatrix} (\{c\},\varnothing,\{c\}) \end{pmatrix}$$

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Is the following RS irreducible?

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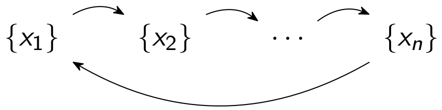
Therefore we need only to study the RSs $\mathcal{RC}_n$ of the form:

$$\begin{array}{c}(\{x_1\},\varnothing,\{x_2\})\\(\{x_2\},\varnothing,\{x_3\})\\\vdots\\(\{x_n\},\varnothing,\{x_1\})\end{array}\Rightarrow$$



Bijective $\mathcal{RS}(1,0)$ irreducibility: $n = 2$

Consider the RS $\mathcal{RC}_2$ over background set: $S = \{a, b\}$.

Reactions:
$$(\{a\}, \varnothing, \{b\})$$
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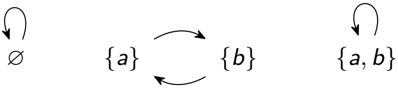

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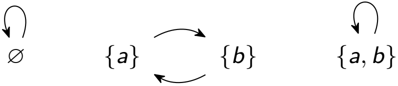
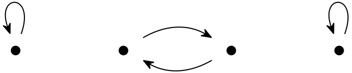


Dynamical system $\mathcal{D}(\mathcal{RC}_2)$:



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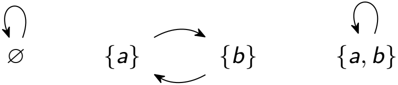

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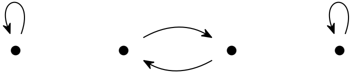

$\mathcal{D}(\mathcal{RC}_2)$ is irreducible.

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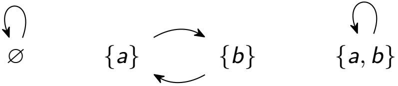

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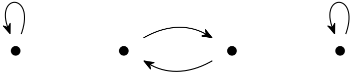

$\mathcal{D}(\mathcal{RC}_2)$ is irreducible.

Lemma

For every $k \geq 1$, $\mathcal{RC}_{2^k}$ is irreducible.

Bijective $\mathcal{RS}(1,0)$ irreducibility: $n = 2$

Proof

The following dynamical system is irreducible:

$$\mathcal{D} = \bullet \circlearrowleft \quad \bullet \circlearrowleft \quad \bullet \rightleftarrows \bullet$$

Suppose there exists $\mathcal{D}_1 \neq \bullet \circlearrowleft$ and $\mathcal{D}_2 \neq \bullet \circlearrowleft$ such that

$$\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2$$

1. $|\mathcal{D}_1| \times |\mathcal{D}_2| = 4$, thus $|\mathcal{D}_1| = |\mathcal{D}_2| = 2$ since both are different from $\bullet \circlearrowleft$.

Bijective $\mathcal{RS}(1,0)$ irreducibility: $n = 2$

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$$\Rightarrow \begin{cases} \mathcal{D}_1 = \bullet\circlearrowleft \ \bullet\circlearrowleft \\ \mathcal{D}_2 = \bullet\circlearrowleft + \mathcal{D}_3 \Rightarrow \mathcal{D}_3 = \bullet\circlearrowleft \end{cases}$$

Bijective $\mathcal{RS}(1,0)$ irreducibility: $n = 2$

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but $\mathcal{D} \neq (\bullet \circlearrowleft \ \bullet \circlearrowleft) \times (\bullet \circlearrowleft \ \bullet \circlearrowleft)$.

Bijective $\mathcal{RS}(1,0)$ irreducibility: $n = 3$

Consider the RS $\mathcal{RC}_3$ over background set: $S = \{a, b, c\}$.

Reactions:

$$(\{a\}, \varnothing, \{b\})$$
$$(\{b\}, \varnothing, \{c\})$$
$$(\{c\}, \varnothing, \{a\})$$

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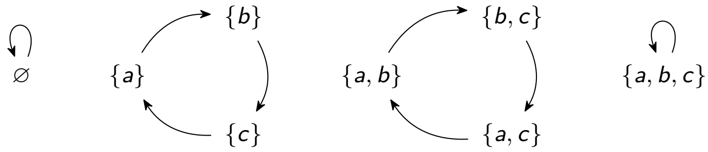
Dynamics of $\mathcal{RC}_3$:



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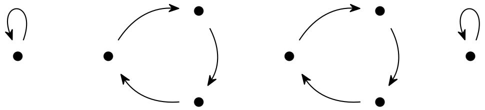

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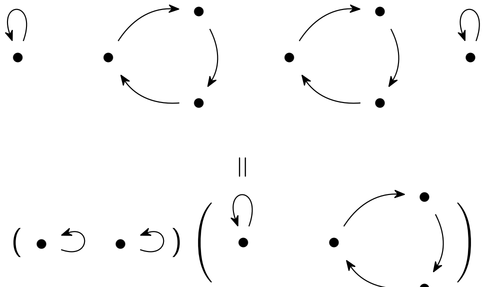
Dynamical system $\mathcal{D}(\mathcal{RC}_3)$:



Bijective $\mathcal{RS}(1,0)$ irreducibility: summary

Lemma

For every $k \geq 1$, $\mathcal{RC}_{2^k}$ is irreducible.

Lemma

If p prime, $p \neq 2$, and $p|n$ then $\mathcal{RC}_n$ is reducible.

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With similar techniques, we can also achieve:

Theorem (Bijective antitone)

Deciding if a given bijective $\mathcal{RS}(0,1)$ is irreducible is in **L**.

Conclusions

This talk

- Irreducibility and Divisibility for succinct representations of the dynamics (e.g. Reaction Systems) are **NP**-hard.
- Irreducibility can be solved very efficiently in some bijective cases.

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- IRREDUCIBILITY and DIVISIBILITY for succinct representations of the dynamics (e.g. Reaction Systems) are **NP**-hard.
- IRREDUCIBILITY can be solved very efficiently in some bijective cases.

Open problems

- Tighter upper bound of IRREDUCIBILITY and DIVISIBILITY for succinct representations of the dynamics.
- Further study on the bijective case.

My research
RSs and others

Ongoing

- Dynamical properties of RSs: complete problems for all levels of the polynomial hierarchy (with Kévin).
- Equations on FDDSs (with Enrico and Marius).
- Majority automata networks and counting complexity (with Pedro and Martin at Adolfo Ibáñez University).

Other topics

- Reaction Automata.
- Genetic Programming and RSs: application to cryptographic functions.
- Fine-grained complexity applied to string matching.

Merci de votre attention!